CARIBBEAN EXAMINATIONS COUNCIL

Caribbean Advanced Proficiency Examination®

CAPE®

PURE MATHEMATICS

Effective for examinations from May–June 2013

Published by the Caribbean Examinations Council

Correspondence related to the syllabus should be addressed to:

The Pro-Registrar
Caribbean Examinations Council
Caenwood Centre
37 Arnold Road, Kingston 5, Jamaica

Telephone Number:+1(876) 630-5200
Facsimile Number: +1(876) 967-4972
E-mail Address: cxcwzo@cxc.org
Website: www.cxc.org

This document CXC A6/U2/12 replaces CXC A6/U2/07 issued in 2007.

Please note that the syllabus has been amended and amendments are indicated by italics.

First issued 1999
Revised 2004
Revised 2007
Amended 2012

Please check the website www.cxc.org for updates on CXC's syllabuses.

Contents

Introduction

The Caribbean Advanced Proficiency Examination (CAPE) is designed to provide certification of the academic, vocational and technical achievement of students in the Caribbean who, having completed a minimum of five years of secondary education, wish to further their studies. The examinations address the skills and knowledge acquired by students under a flexible and articulated system where subjects are organised in 1-Unit or 2-Unit courses with each Unit containing three Modules. Subjects examined under CAPE may be studied concurrently or singly.

The Caribbean Examinations Council offers three types of certification. The first is the award of a certificate showing each CAPE Unit completed. The second is the CAPE Diploma, awarded to candidates who have satisfactorily completed at least six Units including Caribbean Studies. The third is the *CXC* Associate Degree, awarded for the satisfactory completion of a prescribed cluster of seven CAPE Units including Caribbean Studies and Communication Studies. For the CAPE Diploma and the *CXC* Associate Degree, candidates must complete the cluster of required Units within a maximum period of five years.

Recognised educational institutions presenting candidates for CXC Associate Degree in one of the nine categories must, on registering these candidates at the start of the qualifying year, have them confirm in the required form, the Associate Degree they wish to be awarded. Candidates will not be awarded any possible alternatives for which they did not apply.

Pure Mathematics Syllabus

♦ RATIONALE

Mathematics is one of the oldest and most universal means of creating, communicating, connecting and applying structural and quantitative ideas. The discipline of Mathematics allows the formulation and solution of real-world problems as well as the creation of new mathematical ideas, both as an intellectual end in itself, as well as a means to increase the success and generality of mathematical applications. This success can be measured by the quantum leap that occurs in the progress made in other traditional disciplines once mathematics is introduced to describe and analyse the problems studied. It is therefore essential that as many persons as possible be taught not only to be able to use mathematics, but also to understand it.

Students doing this syllabus will have already been exposed to Mathematics in some form mainly through courses that emphasise skills in using mathematics as a tool, rather than giving insight into the underlying concepts. To enable students to gain access to mathematics training at the tertiary level, to equip them with the ability to expand their mathematical knowledge and to make proper use of it, it is necessary that a mathematics course at this level should not only provide them with more advanced mathematical ideas, skills and techniques, but encourage them to understand the concepts involved, why and how they "work" and how they are interconnected. It is also to be hoped that, in this way, students will lose the fear associated with having to learn a multiplicity of seemingly unconnected facts, procedures and formulae. In addition, the course should show them that mathematical concepts lend themselves to generalisations, and that there is enormous scope for applications to solving real problems.

Mathematics covers extremely wide areas. However, students can gain more from a study of carefully selected, representative areas of Mathematics, for a "mathematical" understanding of these areas, rather than a superficial overview of a much wider field. While proper exposure to a mathematical topic does not immediately make students into experts in it, proper exposure will certainly give the students the kind of attitude which will allow them to become experts in other mathematical areas to which they have not been previously exposed. The course is therefore intended to provide quality in selected areas rather than in a large number of topics.

This syllabus will contribute to the development of the Ideal Caribbean Person as articulated by the CARICOM Heads of Government in the following areas: "demonstate multiple literacies, independent and critical thinking and innovative application of science and technology to problem solving. Such a person should also demonstrate a positive work attitude and value and display creative imagination and entrepreneurship".

♦ AIMS

The syllabus aims to:

1. provide understanding of mathematical concepts and structures, their development and the relationships between them;

2. enable the development of skills in the use of mathematical and *information, communication and technology (ICT)* tools;

3. develop an appreciation of the idea of mathematical proof, the internal logical coherence of Mathematics, and its consequent universal applicability;

4. develop the ability to make connections between distinct concepts in Mathematics, and between mathematical ideas and those pertaining to other disciplines;

5. develop a spirit of mathematical curiosity and creativity, as well as a sense of enjoyment;

6. enable the analysis, abstraction and generalisation of mathematical ideas;

7. develop in students the skills of recognising essential aspects of concrete, real-world problems, formulating these problems into relevant and solvable mathematical problems and mathematical modelling;

8. develop the ability of students to carry out independent or group work on tasks involving mathematical modelling;

9. *integrate ICT tools and skills;*

10. provide students with access to more advanced courses in Mathematics and its applications at tertiary institutions.

♦ SKILLS AND ABILITIES TO BE ASSESSED

The assessment will test candidates' skills and abilities in relation to three cognitive levels.

1. **Conceptual knowledge** is the ability to **recall**, **select** and **use** appropriate facts, concepts and principles in a variety of contexts.

2. **Algorithmic knowledge** is the ability to **manipulate** mathematical expressions and procedures using appropriate symbols and language, logical deduction and inferences.

3. **Reasoning** is the ability to **select** appropriate strategy or select, **use** and **evaluate** mathematical models and **interpret** the results of a mathematical solution in terms of a given real-world problem and engage in problem-solving.

♦ PRE-REQUISITES OF THE SYLLABUS

Any person with a **good** grasp of the contents of the syllabus of the Caribbean Secondary Education Certificate (CSEC) General Proficiency course in Mathematics, *and/or the Caribbean Secondary Education Certificate (CSEC) General Proficiency course in Additional Mathematics*, or equivalent, should be able to undertake the course. However, successful participation in the course will also depend on the possession of good verbal and written communication skills.

♦ STRUCTURE OF THE SYLLABUS

The syllabus is arranged into two (2) Units, Unit 1 which will lay the foundation, and Unit 2 which expands on, and applies, the concepts formulated in Unit 1.

It is therefore recommended that Unit 2 be taken after satisfactory completion of Unit 1 or a similar course. Completion of each Unit will be separately certified.

Each Unit consists of three Modules.

Unit 1: Algebra, Geometry and Calculus, contains three Modules each requiring approximately 50 hours. The total teaching time is therefore approximately 150 hours.

Module 1 - Basic Algebra and Functions
Module 2 - *Trigonometry,* Geometry and Vectors
Module 3 - Calculus I

Unit 2: *Complex Numbers,* Analysis and Matrices, contains three Modules, each requiring approximately 50 hours. The total teaching time is therefore approximately 150 hours.

Module 1 - *Complex Numbers and Calculus II*
Module 2 - Sequences, Series and Approximations
Module 3 - *Counting, Matrices and Differential Equations*

♦ RECOMMENDED 2-UNIT OPTIONS

1. Pure Mathematics Unit 1 **AND** Pure Mathematics Unit 2.

2. Applied Mathematics Unit 1 **AND** Applied Mathematics Unit 2.

3. Pure Mathematics Unit 1 **AND** Applied Mathematics Unit 2.

♦ MATHEMATICAL MODELLING

Mathematical Modelling should be used in both Units 1 and 2 to solve real-world problems.

A. The topic Mathematical Modelling involves the following steps:

   1. identification of a real-world situation to which modelling is applicable;

   2. carry out the modelling process for a chosen situation to which modelling is applicable;

   3. discuss and evaluate the findings of a mathematical model in a written report.

B. The Modelling process requires:

   1. a clear statement posed in a real-world situation, and identification of its essential features;

   2. translation or abstraction of the problem, giving a representation of the essential features of the real-world;

   3. solution of the mathematical problem (analytic, numerical, approximate);

   4. testing the appropriateness and the accuracy of the solution against behaviour in the real-world;

   5. refinement of the model as necessary.

C. Consider the two situations given below.

   1. A weather forecaster needs to be able to calculate the possible effects of atmospheric pressure changes on temperature.

   2. An economic adviser to the Central Bank Governor needs to be able to calculate the likely effect on the employment rate of altering the Central Bank’s interest rate.

   In each case, people are expected to predict something that is likely to happen in the future. Furthermore, in each instance, these persons may save lives, time, and money or change their actions in some way as a result of their predictions.

One method of predicting is to set up a mathematical model of the situation. A mathematical model is not usually a model in the sense of a scale model motor car. A mathematical model is a way of describing an underlying situation mathematically, perhaps with equations, with rules or with diagrams.

D. Some examples of mathematical models are:

1. Equations

(a) Business

A recording studio invests \$25 000 to produce a master CD of a singing group. It costs \$50.00 to make each copy from the master and cover the operating expenses. We can model this situation by the equation or mathematical model,

$$C = 50.00\,x + 25\,000$$

where C is the cost of producing x CDs. With this model, one can predict the cost of producing 60 CDs or 6 000 CDs.

Is the equation $x + 2 = 5$ a mathematical model? Justify your answer.

(b) Banking

Suppose you invest \$100.00 with a commercial bank which pays interest at 12% per annum. You may leave the interest in the account to accumulate. The equation $A = 100(1.12)^n$ can be used to model the amount of money in your account after n years.

2. Table of Values

Traffic Management

The table below shows the safe stopping distances for cars recommended by the Highway Code.

Speed ***m/h***	**Thinking Distance m**	**Braking Distance m**	**Overall Stopping Distance m**
20	6	6	12
30	9	14	23
40	12	24	36
50	15	38	53
60	18	55	73
70	21	75	96

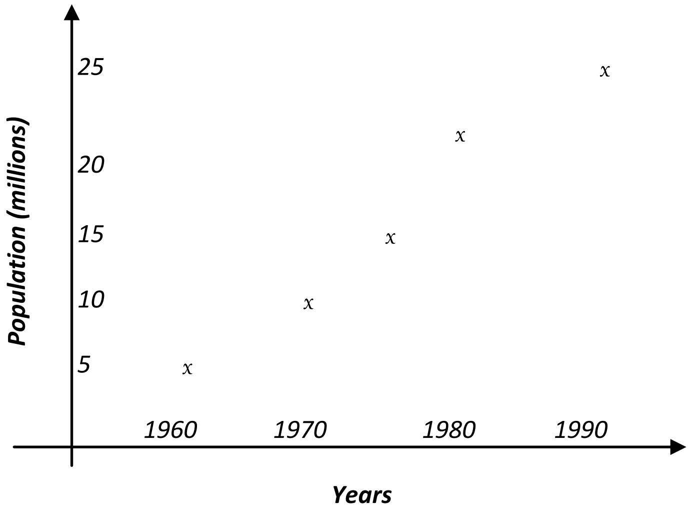
We can predict our stopping distance when travelling at 50 m/h from this model.

3. Rules of Thumb

You might have used some mathematical models of your own without realising it; perhaps you think of them as “rules of thumb”. For example, in the baking of hams, most cooks use the rule of thumb that “bake ham fat side up in roasting pan in a moderate oven (160ºC) ensuring 25 to 40 minutes per ½ kg”. The cook is able to predict how long it takes to bake his ham without burning it.

4. Graphs

Not all models are symbolic in nature; they may be graphical. For example, the graph below shows the population at different years for a certain country.



RESOURCE

Hartzler, J. S. and Swetz, F. — *Mathematical Modelling in the Secondary School Curriculum, A Resource Guide of Classroom Exercises,* Vancouver, United States of America: National Council of Teachers of Mathematics, Incorporated, Reston, 1991.

♦ UNIT 1: ALGEBRA, GEOMETRY AND CALCULUS

MODULE 1: BASIC ALGEBRA AND FUNCTIONS

GENERAL OBJECTIVES

On completion of this Module, students should:

1. *develop* the ability to construct simple proofs of mathematical assertions;

2. *understand* the concept of a function;

3. be confident in the manipulation of algebraic expressions and the solutions of equations and inequalities;

4. *understand the properties and significance of the exponential and logarithm functions;*

5. *develop* the ability to use concepts to model and solve real-world problems.

SPECIFIC OBJECTIVES

(A) Reasoning *and Logic*

Students should be able to:

1. *identify simple and compound propositions;*

2. *establish the truth value of compound statements using truth tables;*

3. *state the converse, contrapositive and inverse of a conditional (implication) statement;*

4. *determine whether two statements are logically equivalent.*

CONTENT

(A) *Reasoning and Logic*

1. *Simple statement (proposition), connectives (conjuction, disjunction, negation, conditional, bi-conditional), compound statements.*

2. *Truth tables.*

3. *Converse and contrapositive of statements.*

4. *Logical equivalence.*

5. *Identities involving propositions.*

UNIT 1
MODULE 1: BASIC ALGEBRA AND FUNCTIONS (cont'd)

SPECIFIC OBJECTIVES

(B) The Real Number System – $\mathbb{R}$

Students should be able to:

1. *perform binary operations;*
2. *use the concepts of identity, closure, inverse, commutativity, associativity, distributivity addition, multiplication and other simple binary operations;*
3. *perform operations involving surds;*
4. construct simple proofs, specifically direct proofs, or proof by the use of counter examples;
5. *use the summation notation ($\sum$);*
6. establish simple proofs by using the principle of mathematical induction.

CONTENT

(B) The Real Number System – $\mathbb{R}$

1. *Definition of binary operation.*
2. *Applications of the concepts of commutativity, associativity, distributivity, identity, inverse and closure.*
3. Axioms of the system - including commutative, associative and distributive laws; non-existence of the multiplicative inverse of zero.
4. Methods of proof - direct, counter-examples.
5. Simple applications of mathematical induction.

SPECIFIC OBJECTIVES

(C) *Algebraic Operations*

Students should be able to:

1. *apply the Remainder Theorem;*
2. use the Factor Theorem to find factors and to evaluate unknown coefficients;
3. extract all factors of $a^n - b^n$ for positive integers $n \leq 6$;
4. use the concept of identity of polynomial expressions.

CONTENT

(C) Algebraic Operations

1. Factor Theorem.

2. Remainder Theorem.

SPECIFIC OBJECTIVES

(D) *Exponential and Logarithmic Functions*

Students should be able to:

1. *define an exponential function* $y = a^x$ *for* $a \in \mathbb{R}$;

2. *sketch the graph of* $y = a^x$*;*

3. *define a logarithmic function as the inverse of an exponential function;*

4. *define the exponential functions* $y = e^x$ *and its inverse* $y = \ln x$, *where* $\ln x \equiv \log_e x$*;*

5. *use the fact that* $y = \ln x \Leftrightarrow x = e^y$*;*

6. *simplify expressions by using laws of logarithms;*

7. *use logarithms to solve equations of the form* $a^x = b$*;*

8. solve problems involving changing of the base of a logarithm.

CONTENT

(D) Exponential and Logarithmic Functions

1. Graphs of the functions a^x and $\log_a x$.

2. Properties of the exponential and logarithmic functions.

3. Exponential and natural logarithmic functions and their graphs.

4. Laws of logarithms applied to problems:

   (a) $\ln (PQ) = \ln P + \ln Q$;

   (b) $\ln (P/Q) = \ln P - \ln Q$;

   (c) $\ln P^a = a \ln P$.

SPECIFIC OBJECTIVES

(E) Functions

Students should be able to:

1. *define mathematically the terms: function, domain, range, one-to-one function (injective function), onto function (surjective function), many-to-one, one-to-one and onto function (bijective function), composition and inverse of functions;*

2. *prove whether or not a given simple function is one-to-one or onto and if its inverse exists;*

3. *use the fact that a function may be defined as a set of ordered pairs;*

4. *use the fact that if g is the inverse function of f, then $\mathrm{f}\,[\mathrm{g}\,(x)] = x$, for all x, in the domain of g;*

5. illustrate by means of graphs, the relationship between the function $y = \mathrm{f}\,(x)$ given in graphical form and $y = |\mathrm{f}\,(x)|$ and the inverse of $\mathrm{f}\,(x)$, that is, $y = \mathrm{f}^{-1}(x)$.

CONTENT

(E) Functions

1. Domain, range, *many-to-one,* composition.

2. Injective, surjective, bijective functions, inverse function.

3. Transformation of the graph $y = \mathrm{f}\,(x)$ to $y = |\mathrm{f}\,(x)|$ and, if appropriate, to $y = \mathrm{f}^{-1}(x)$.

SPECIFIC OBJECTIVES

(F) ***The Modulus Function***

Students should be able to:

1. *define the modulus function;*

2. *use the properties:*

   (a) $|x|$ *is the positive square root of* x^2,

   (b) $|x| < |y|$ *if, and only if,* $x^2 < y^2$,

   (c) $|x| < |y| \Leftrightarrow iff - y < x < y$,

   (d) $|x + y| \leq |x| + |y|$;

3. *solve equations and inequalities involving the modulus function, using algebraic or graphical methods.*

CONTENT

(F) ***The Modulus Function***

1. Definition of the modulus function.

$$|x| = \left\{ \begin{array}{r} x \text{ if } x \geq 0 \\ -x \text{ if } x < 0 \end{array} \right\};$$

2. Equations and inequalities involving simple rational and modulus functions.

SPECIFIC OBJECTIVES

(G) ***Cubic Functions and*** **Equations**

Students should be able to use the relationship between the *sum of the roots, the product of the roots, the sum of the product of the roots pair-wise* and the coefficients *of* $ax^3 + bx^2 + cx + d = 0$.

CONTENT

(G) ***Cubic Functions and Equations***

Sums and products, with applications, of the roots *of* cubic equations.

Suggested Teaching and Learning Activities

To facilitate students' attainment of the objectives of this Module, teachers are advised to engage students in the teaching and learning activities listed below.

1. The Real Number System

(a) *The teacher should do a brief review of the number system before starting this section.*

(b) *The teacher should encourage students to practice different methods of proof, for example, to prove that the product of two consecutive integers is an even integer.*

2. Proof by Mathematical Induction (MI)

Typical Question

Prove that some formula or statement P is true for all positive integers $n \geq k$, where k is some positive integer; usually $k = 1$.

Procedure

Step 1: Verify that when $k = 1$: P is true for $n = k = 1$. This establishes that P is true for $n = 1$.

Step 2: Assume P is true for $n = k$, where k is a positive integer > 1. At this point, the statement k replaces n in the statement P and is taken as true.

Step 3: Show that P is true for $n = k + 1$ using the true statement in step 2 with n replaced by k.

Step 4: At the end of step 3, it is stated that statement P is true for all positive integers $n \geq k$.

Summary

Proof by MI: For $k > 1$, verify Step 1 for k and proceed through to Step 4.

Observation

Most users of MI do not see how this proves that P is true. The reason for this is that there is a massive gap between Steps 3 and 4 which can only be filled by becoming aware that Step 4 only follows because Steps 1 to 3 are repeated an infinity of times to generate the set of all positive integers. The focal point is the few words "for all positive integers $n \geq k$" which points to the determination of the set S of all positive integers for which P is true.

Step 1 says that $1 \in S$ for $k = 1$.
Step 3 says that $k + 1 \in S$ whenever $k \in S$, so immediately $2 \in S$ since $1 \in S$.

Iterating on Step 3 says that $3 \in S$ since $2 \in S$ and so on, so that $S = \{1, 2, 3 ...\}$, that is, S is the set of all positive integers when $k = 1$ which brings us to Step 4.

When $k > 1$, the procedure starts at a different positive integer, but the execution of steps is the same. Thus, it is necessary to explain what happens between Steps 3 and 4 to obtain a full appreciation of the method.

Example 1: Use Mathematical Induction to prove that $n^3 - n$ is divisible by 3, whenever n is a positive integer.

Solution: Let P (n) be the proposition that "$n^3 - n$ is divisible by 3".

Basic Step: P (1) is true, since $1^3 - 1 = 0$ which is divisible by 3.

Inductive Step: Assume P (n) is true: that is, $n^3 - n$ is divisible by 3.
We must show that P (n + 1) is true, if P(n) is true. That is, is, $(n + 1)^3 - (n + 1)$ is divisible by 3.

$$\text{Now, } (n+1)^3 - (n + 1) = (n^3 + 3n^2 + 3n + 1) - (n + 1)$$
$$= (n^3 - n) + 3(n^2 + n)$$

Both terms are divisible by 3 since $(n^3 - n)$ is divisible by 3 by the assumption and $3(n^2 + n)$ is a multiple of 3. Hence, P (n + 1) is true whenever P (n) is true.
Thus, $n^3 - n$ is divisible by 3 whenever n is a positive integer.

Example 2: Prove by Mathematical Induction that the sum S_n of the first n odd positive integers is n^2.

Solution: Let $P(n)$ be the proposition that the sum S_n of the first n odd positive integers is n^2.

Basic Step: For $n = 1$ the first odd positive integer is 1, so $S_1 = 1$, that is: $S_1 = 1 = 1^2$, hence $P(1)$ is true.

Inductive Step: Assume $P(n)$ is true. That is, $S_n = 1 + 3 + 5 + \dots + (2n - 1) = n^2$.

$$\begin{aligned} \text{Now, } S_{n+1} &= 1 + 3 + 5 + \dots + (2n - 1) + (2n + 1) \\ &= [1 + 3 + 5 + \dots + (2n - 1)] + (2n + 1) \\ &= n^2 + (2n + 1), \text{ by the assumption,} \\ &= (n + 1)^2 \end{aligned}$$

Thus, $P(n+1)$ is true whenever $P(n)$ is true.

Since $P(1)$ is true and $P(n) \rightarrow P(n + 1)$, the proposition $P(n)$ is true for all positive integers n.

3. Functions (Injective, surjective, bijective) – Inverse Function

Mathematical proof that a function is one-to-one (injective), onto (surjective) or (one-to-one and onto function) bijective should be introduced at this stage.

Teacher and students should explore the mapping properties of quadratic functions which:

(a) will, or will not, be injective, depending on which subset of the real line is chosen as the domain;

(b) will be surjective if its range is taken as the co-domain (completion of the square is useful here);

(c) if both injective and surjective, will have an inverse function which can be constructed by solving a quadratic equation.

Example: Use the function $f : A \rightarrow B$ given by $f(x) = 3x^2 + 6x + 5$, where the domain A is alternatively the whole of the real line, or the set $\{x \in \mathbb{R} \mid x \geq -1\}$, and the co-domain B is $\mathbb{R}$ or the set $\{y \in \mathbb{R} \mid y \geq 2\}$.

4. *Cubic Equations*

Teachers should first review the theory of the quadratic equation and the nature of its roots.

RESOURCES

Bostock, L. and Chandler, S.	*Core Mathematics for A-Levels*, United Kingdom: Stanley Thornes Publishing Limited, 1997.
Campbell, E.	*Pure Mathematics for CAPE, Vol. 1,* Jamaica: LMH Publishing Limited, 2007.
Caribbean Examinations Council	*Injective and Surjective Functions*, Barbados: 1998.
Caribbean Examinations Council	*Proof by Mathematical Induction (MI), Barbados: 2004.*
Caribbean Examinations Council	*The Real Number System*, Barbados: 1997.
Greaves, Y.	*Solution of Simultaneous Linear Equations by Row Reduction,* Barbados: Caribbean Examinations Council, 1998.
Hartzler, J. and Swetz, F.	*Mathematical Modelling in the Secondary School Curriculum, A Resource Guide of Classroom Exercises,* Vancouver, United States of America: National Council of Teachers of Mathematics, Incorporated Reston*, 1991*.
Martin, A., Brown, K., Rigby, P. and Ridley, S.	*Advanced Level Mathematics Tutorials Pure Mathematics CD-ROM sample (Trade Edition),* Cheltenham, United Kingdom: Stanley Thornes (Publishers) Limited, Multi-user version and Single-user version, 2000.

UNIT 1

MODULE 2: *TRIGONOMETRY,* GEOMETRY AND VECTORS

GENERAL OBJECTIVES

On completion of this Module, students should:

1. *develop* the ability to represent and deal with objects in two and three dimensions through the use of coordinate geometry and vectors;

2. *develop* the ability to manipulate and describe the behaviour of trigonometric functions;

3. *develop* the ability to establish trigonometric identities;

4. acquire the skills to solve trigonometric equations;

5. *acquire the skills to conceptualise and to manipulate objects in two and three dimensions;*

6. *develop* the ability to use concepts to model and solve real-world problems.

SPECIFIC OBJECTIVES

(A) Trigonometric Functions, Identities and Equations (all angles will be assumed to be in radians unless otherwise stated)

Students should be able to*:*

1. *use compound-angle formulae;*

2. *use the reciprocal functions of* sec x, cosec x *and* cot x*;*

3. *derive identities for the following:*

   (a) sin kA, cos kA, tan kA, *for* $k \in \mathbb{Q}$,

   (b) $\tan^2 x$, $\cot^2 x$, $\sec^2 x$ and $\text{cosec}^2 x$,

   (c) sin A ± sin B, cos A ± cos B;

4. *prove further identities using Specific Objective 3;*

5. express $a\cos\theta + b\sin\theta$ in the form $r\cos(\theta \pm \alpha)$ and $r\sin(\theta \pm \alpha)$, where r is positive, $0 < \alpha < \frac{\pi}{2}$;

6. find the general solution of equations of the form:

   (a) sin $k\theta = s$,

   (b) cos $k\theta = c$,

   (c) tan $k\theta = t$,

   (d) a cos θ + b sin θ = c,
   for $a, b, c, k, s, t, \in \mathbb{R}$;

7. *find the solutions of the equations in Specific Objectives* 6 *above for a given range;*

8. obtain maximum or minimum values of $\mathrm{f}\,(a \cos \theta + b \sin \theta)$ for $0 \leq \theta \leq 2\pi$.

CONTENT

(A) Trigonometric Functions, Identities and Equations (all angles will be assumed to be radians)

1. The functions cot x, sec x, cosec x.

2. Compound-angle formulae for sin $(A \pm B)$, cos $(A \pm B)$, tan $(A \pm B)$.

3. *Multiple-angle formulae*.

4. Formulae for sin A ± sin B, cos A ± cos B.

5. *Expression of a* cos θ + b sin θ *in the forms* r sin $(\theta \pm \alpha)$ and r cos $(\theta \pm \alpha)$, *where r is positive,* $0 < \alpha < \dfrac{\pi}{2}$.

6. General solutions of simple trigonometric equations.

7. Trigonometric identities $\cos^2\theta + \sin^2\theta \equiv 1$, $1 + \cot^2\theta \equiv \operatorname{cosec}^2\theta$, $1 + \tan^2\theta \equiv \sec^2\theta$.

8. Maximum and minimum values of functions of sin θ and cos θ.

SPECIFIC OBJECTIVES

(B) Co-ordinate Geometry

Students should be able to:

1. find equations of tangents and normals to circles;
2. find the points of intersection of a curve with a straight line;
3. find the points of intersection of two curves;
4. obtain the Cartesian equation of a curve given its parametric representation;
5. obtain the parametric representation of a curve given its Cartesian equation;
6. *determine the loci of points satisfying given properties*.

CONTENT

(B) Co-ordinate Geometry

1. Properties of the circle.
2. Tangents and normals.
3. Intersections between lines and curves.
4. Cartesian equations and parametric representations of curves including the parabola and ellipse.
5. *Loci.*

SPECIFIC OBJECTIVES

(C) Vectors

Students should be able to:

1. *express a vector in the form* $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ *or* $x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$

   where **i***,* **j** *and* **k** *are unit vectors in directions of* x-,y- *and* z-*axis respectively;*
2. define equality of two vectors;
3. add and subtract vectors;
4. multiply a vector by a scalar quantity;

5. *derive and use unit vectors, position vectors and displacement vectors;*

6. find the magnitude and direction of a vector;

7. *find the angle between two given vectors using scalar product;*

8. *find the equation of a line in vector form, parametric form, Cartesian form, given a point on the line and a vector parallel to the line;*

9. *determine whether two lines are parallel, intersecting, or skewed;*

10. *find the equation of the plane, in the form* $x\mathbf{i} + y\mathbf{j} + z\mathbf{k} = d$, $\mathbf{r.n} = d$, *given a point in the plane and the normal to the plane.*

CONTENT

(C) Vectors

1. Expression of a given vector in the form $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ or $x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$.

2. Equality, addition and subtraction of vectors; multiplication by a scalar.

3. Position vectors, unit vectors, displacement vectors.

4. Length (magnitude/modulus) and direction of a vector.

5. Scalar (Dot) Product.

6. *Vector equation of a line.*

7. *Equation of a plane.*

<u>**Suggested Teaching and Learning Activities**</u>

To facilitate students' attainment of the objectives of this Module, teachers are advised to engage students in the teaching and learning activities listed below.

1. Trigonometric Identities

Much practice is required to master proofs of Trigonometric Identities using identities such as the formulae for:

sin $(A \pm B)$, cos $(A \pm B)$, tan $(A \pm B)$, sin $2A$, cos $2A$, tan $2A$

Example: The identity $\dfrac{1-\cos 4\theta}{\sin 4\theta} \equiv \tan 2\theta$ can be established by realising that $\cos 4\theta \equiv 1 - 2\sin^2 2\theta$ and $\sin 4\theta \equiv 2\sin 2\theta \cos 2\theta$.

Derive the trigonometric functions sin x *and* cos x *for angles* x *of any value (including negative values), using the coordinates of points on the unit circle.*

2. Vectors

Teachers should introduce students to the three dimensional axis and understand how to plot vectors in three dimensions.

RESOURCE

Bostock, L. and Chandler, S. — *Mathematics - The Core Course for A-Level,* United Kingdom: Stanley Thornes (Publishers) Limited, 1997.

Campbell, E. — *Pure Mathematics for CAPE, Vol. 1, Jamaica:* LMH Publishing Limited, 2007.

UNIT 1
MODULE 3: CALCULUS I

GENERAL OBJECTIVES

On completion of this Module, students should:

1. *understand the concept of continuity of a function from its graph;*

2. *develop* the ability to find the limits (when they exist) of functions in simple cases;

3. know the relationships between the derivative of a function at a point and the behaviour of the function and its tangent at that point;

4. know the relationship between integration and differentiation;

5. know the relationship between integration and the area under the graph of the function;

6. *develop* the ability to use concepts to model and solve real-world problems.

SPECIFIC OBJECTIVES

(A) Limits

Students should be able to:

1. use graphs to determine the continuity and discontinuity of functions;

2. describe the behaviour of a function f (x) as x gets arbitrarily close to some given fixed number, using a descriptive approach;

3. use the limit notation $\lim_{x \to a} \mathrm{f}(x) = L$, $\mathrm{f}(x) \to \mathrm{L}$ as $x \to a$;

4. *use the simple limit theorems*:
   If $\lim_{x \to a} \mathrm{f}(x) = F$, $\lim_{x \to a} \mathrm{g}(x) = G$ and k is a constant,

   then $\lim_{x \to a} k\mathrm{f}(x) = kF$, $\lim_{x \to a} \mathrm{f}(x)\mathrm{g}(x) = FG$, $\lim_{x \to a} \{\mathrm{f}(x) + \mathrm{g}(x)\} = F + G$,

   and, provided $G \neq 0$, $\lim_{x \to a} \dfrac{\mathrm{f}(x)}{\mathrm{g}(x)} = \dfrac{F}{G}$;

5. *use limit theorems in simple problems;*

6. *use the fact that* $\lim_{x \to 0} \dfrac{\sin x}{x} = 1$, *demonstrated by a geometric approach;*

7. *identify the point(s) for which a function is (un)defined;*

8. identify the points for which a function is continuous;

9. *identify the point(s) where a function is discontinuous;*

10. *use the concept of left-handed or right-handed limit, and continuity.*

CONTENT

(A) Limits

1. Concept of limit of a function.

2. Limit Theorems.

3. Continuity and Discontinuity.

SPECIFIC OBJECTIVES

(B) Differentiation I

Students should be able to:

1. *define the derivative of a function at a point as a limit;*

2. *differentiate, from first principles, functions such as*:

   (a) $f(x) = k$ *where* $k \in \mathbb{R}$,

   (b) $f(x) = x^n$, *where* $n \in \{-3, -2, -1, -½, ½, 1, 2, 3\}$,

   (c) $f(x) = \sin x$,

   (d) $f(x) = \cos x$;

3. use the *sum*, product and quotient rules for differentiation;

4. differentiate *sums*, products and quotients of:

   (a) polynomials,

   (b) trigonometric functions*;*

5. *apply the chain rule in the differentiation of*

   (a) *composite functions (substitution),*

   (b) *functions given by parametric equations;*

6. *solve problems involving rates of change;*

7. use the sign of the derivative to investigate where a function is increasing or decreasing;

8. *apply the concept of stationary (critical) points;*

9. calculate second derivatives;

10. interpret the significance of the sign of the second derivative;

11. use the sign of the second derivative to determine the nature of stationary points;

12. *sketch graphs of polynomials, rational functions and trigonometric functions using the features of the function and its first and second derivatives (including horizontal and vertical asymptotes);*

13. describe the behaviour of such graphs for large values of the independent variable;

14. obtain equations of tangents and normals to curves.

CONTENT

(B) Differentiation I

1. The Gradient.

2. The Derivative as a limit.

3. Rates of change.

4. Differentiation from first principles.

5. Differentiation of simple functions, product, quotients.

6. *Stationary points, the chain rule and parametric equations*.

7. Second derivatives of functions.

8. Curve sketching.

9. Tangents and Normals to curves.

UNIT 1
MODULE 3: CALCULUS I (cont'd)

SPECIFIC OBJECTIVES

(C) Integration I

Students should be able to:

1. *recognise integration as the reverse process of differentiation;*

2. demonstrate an understanding of the indefinite integral and the use of the integration notation $\int f(x)\, dx$;

3. show that the indefinite integral represents a family of functions which differ by constants;

4. demonstrate use of the following integration theorems:

   (a) $\int cf(x)dx = c\int f(x)dx$, where c is a constant,

   (b) $\int \{f(x) \pm g(x)\}\, dx = \int f(x)\, dx \pm \int g(x)\, dx$;

5. find:

   (a) indefinite integrals using integration theorems,

   (b) integrals of polynomial functions,

   (c) integrals of simple trigonometric functions;

6. integrate *using substitution;*

7. use the results:

   (a) $\int_a^b f(x)dx = \int_a^b f(t)dt$,

   (b) $\int_0^a f(x)dx = \int_0^a f(x-a)dx$ for $a > 0$,

   (c) $\int_a^b f(x)\, dx = F(b) - F(a)$, where $F'(x) = f(x)$;

8. apply *integration to:*

   (a) finding areas under the curve,

   (b) finding areas between two curves,

   (c) *finding volumes of revolution by rotating regions about both the x- and y-axes*;

9. *given a rate of change with or without initial boundary conditions:*

   (a) *formulate a differential equation of the form $y' = \mathrm{f}(x)$ or $y'' = \mathrm{f}(x)$ where f is a polynomial or a trigonometric function,*

   (b) *solve the resulting differential equation in* (a) *above and interpret the solution where applicable.*

CONTENT

(c) Integration I

1. Integration as the reverse of differentiation.

2. Linearity of integration.

3. Indefinite integrals (concept and use).

4. Definite integrals.

5. Applications of integration – areas, volumes and solutions to elementary differential equations.

6. Integration of polynomials.

7. Integration of simple trigonometric functions.

8. Use of $\int_a^b \mathrm{f}(x)\,\mathrm{d}x = \mathrm{F}(b) - \mathrm{F}(a)$, where $\mathrm{F}'(x) = \mathrm{f}(x)$.

9. *Simple first or second* order differential equations *of the type* $y' = \mathrm{f}(x)$ or $y'' = \mathrm{f}(x)$.

Suggested Teaching and Learning Activities

To facilitate students' attainment of the objectives of this Module, teachers are advised to engage students in the teaching and learning activities listed below.

The Area under the Graph of a Continuous Function

Class discussion should play a major role in dealing with this topic. Activities such as that which follows may be performed to motivate the discussion.

Example of classroom activity:

Consider a triangle of area equal to $\frac{1}{2}$ units, bounded by the graphs of $y = x$, $y = 0$ and $x = 1$.

(a) Sketch the graphs and identify the triangular region enclosed.

(b) Subdivide the interval [0, 1] into n equal subintervals.

(c) Evaluate the sum, $s(n)$, of the areas of the inscribed rectangles and $S(n)$, of the circumscribed rectangles, erected on each subinterval.

(d) By using different values of n, for example, for n = 5, 10, 25, 50, 100, show that both $s(n)$ and $S(n)$ get closer to the required area of the given region.

RESOURCES

Bostock, L., and Chandler, S.	*Mathematics - The Core Course for A-Level, United Kingdom:* Stanley Thornes Publishing Limited, (Chapters 5, 8 and 9), 1991.
Campbell, E.	*Pure Mathematics for CAPE, Vol. 1, Jamaica:* LMH Publishing Limited, 2007.
Caribbean Examinations Council	*Area under the Graph of a Continuous Function*, Barbados: 1998.
Caribbean Examinations Council	*Differentiation from First Principles: The Power Function, Barbados:* 1998.

♦ UNIT 2: *COMPLEX NUMBERS*, ANALYSIS AND MATRICES

MODULE 1: *COMPLEX NUMBERS AND CALCULUS II*

GENERAL OBJECTIVES

On completion of this Module, students should:

1. *develop the ability to represent and deal with objects in the plane through the use of complex numbers;*

2. *be confident in using the techniques of differentiation and integration;*

3. *develop* the ability to use concepts to model and solve real-world problems.

SPECIFIC OBJECTIVES

(A) Complex Numbers

Students should be able to:

1. recognise the need to use complex numbers to find the roots of the general quadratic equation $ax^2 + bx + c = 0$, when $b^2 - 4ac < 0$;

2. *use the concept that complex roots of equations with constant coefficients occur in conjugate pairs*;

3. write the roots of the equation in that case and relate the sums and products to a, b and c;

4. calculate the square root of a complex number;

5. express complex numbers in the form $a + b\text{i}$ where a, b are real numbers, and identify the real and imaginary parts;

6. add, subtract, multiply and divide complex numbers in the form $a + b\text{i}$, where a and b are real numbers;

7. find the principal value of the argument θ of a non-zero complex number, where $-\pi < \theta \leq \pi$;

8. find the modulus and conjugate of a given complex number;

9. interpret modulus and argument of complex numbers on the Argand diagram;

10. represent complex numbers, their sums, differences and products on an Argand diagram;

11. find the set of all points z (locus of z) on the Argand Diagram such that z satisfies given properties;

12. apply De Moivre's theorem for integral values of n;

13. *use* $e^{ix} = \cos x + i \sin x$, *for real* x.

CONTENT

(A) Complex Numbers

1. Nature of roots of a quadratic equation, sums and products of roots.

2. *Conjugate pairs of complex roots.*

3. Addition, subtraction, multiplication and division of complex numbers in the form $a + b\text{i}$ where a, b are the real and imaginary parts, respectively, of the complex number.

4. The modulus, argument and conjugate of a complex number.

5. Representation of complex numbers on an Argand diagram.

6. Locus of a point.

7. De Moivre's theorem for integral n.

8. *Polar-argument and exponential forms of complex numbers*.

SPECIFIC OBJECTIVES

(B) Differentiation II

Students should be able to:

1. find the derivative of $e^{f(x)}$, where f (x) is a differentiable function of x;

2. find the derivative of ln f (x) (to include functions of x – polynomials or trigonometric);

3. apply the chain rule to obtain gradients *and equations* of tangents and normals to curves given by their parametric equations;

4. use the concept of implicit differentiation, with the assumption that one of the variables is a function of the other;

5. differentiate any combinations of polynomials, trigonometric, exponential and logarithmic functions;

6. differentiate inverse trigonometric functions;

7. obtain second derivatives, $f''(x)$, of the functions in 3, 4, 5 above;

8. *find the first and second partial derivatives of $u = f(x, y)$.*

CONTENT

(B) Differentiation II

1. Application of the chain rule to differentiation.

2. Chain rule and differentiation of composite functions.

3. Gradients of tangents and normals.

4. Implicit differentiation.

5. First derivative of a function which is defined parametrically.

6. Differentiation of inverse trigonometric functions.

7. Differentiation of combinations of functions.

8. Second derivative, that is, $f''(x)$.

9. *First partial derivative.*

10. *Second partial derivative.*

SPECIFIC OBJECTIVES

(C) Integration II

Students should be able to:

1. express a rational function ***(proper and improper)*** in partial fractions in the cases where the denominators are:

   (a) distinct linear factors,

   (b) repeated linear factors,

(c) quadratic factors,

(d) repeated quadratic factors,

(e) combinations of (a) to (d) above (repeated factors will not exceed power 2);

2. express an improper rational function as a sum of a polynomial and partial fractions;

3. integrate rational functions in Specific Objectives 1 and 2 above;

4. integrate trigonometric functions using appropriate trigonometric identities;

5. integrate exponential functions and logarithmic functions;

6. find integrals of the form $\int \frac{f'(x)}{f(x)}\,dx$;

7. use substitutions to integrate functions (the substitution will be given in all but the most simple cases);

8. use integration by parts for combinations of functions;

9. integrate inverse trigonometric functions;

10. derive and use reduction formulae to obtain integrals;

11. use the trapezium rule as an approximation method for evaluating the area under the graph of the function.

CONTENT

(C) Integration II

1. Partial fractions.

2. Integration of rational functions, using partial fractions.

3. Integration by substitution.

4. Integration by parts.

5. Integration of inverse trigonometric functions.

6. Integration by reduction formula.

7. Area under the graph of a continuous function (Trapezium Rule).

Suggested Teaching and Learning Activities

To facilitate students' attainment of the objectives of this Module, teachers are advised to engage students in the teaching and learning activities listed below.

Principal Argument of a Complex Number

The representation of the complex number $z = 1 + \mathrm{i}$ *on the Argand diagram may be used to introduce this topic. Encourage students to indicate and evaluate the argument of* z. *The students' answers should be displayed on the chalkboard.*

Indicate that the location of z *on the Argand diagram is unique, and therefore only one value of the argument is needed to position* z. *That argument is called the principal argument, arg* z, *where:*

$$-\pi < \text{principal argument} \leq \pi.$$

Students should be encouraged to calculate the principal argument by either solving:

(a) *the simultaneous equations*

$$\cos\theta = \frac{\text{Re}(z)}{|z|} \text{ and } \sin\theta = \frac{\text{Im}(z)}{|z|}, \text{with} -\pi < \theta \leq \pi\ ;$$

or,

(b) *the equation*

$$\tan\theta = \frac{\text{Im}(z)}{\text{Re}(z)} \text{ for } \text{Re}(z) \neq 0 \text{ and } -\pi < \theta \leq \pi,$$

together with the representation of z on the Argand diagram.

(c) *Students should be encouraged to find the loci of z-satisfying equations such as:*

(i) $|z - a| = k$;
(ii) $|z - c| = |z - b|$;
(iii) $\arg(z - a) = \alpha$.

RESOURCES

Bostock, L. and Chandler, S. — *Core Mathematics for A-Levels*, United Kingdom: Stanley Thornes Publishing Limited, 1997.

Bradie, B. — *Rate of Change of Exponential Functions*: A Precalculus Perspective, Mathematics Teacher Vol. 91(3), p. 224 – 237.

Campbell, E. *Pure Mathematics for CAPE, Vol. 2, Jamaica:* LMH Publishing Limited, 2007.

Caribbean Examinations Council *The Exponential and Logarithmic Functions – An Investigation, Barbados:* 1998.

Martin, A., Brown, K., Rigby, P. and Ridley, S. Pure Mathematics, *Cheltenham, United Kingdom*: *Stanley Thornes (Publishers) Limited, 2000.*

UNIT 2
MODULE 2: SEQUENCES, SERIES AND APPROXIMATIONS

GENERAL OBJECTIVES

On completion of this Module, students should:

1. *understand* the concept of a sequence as a function from the natural numbers to the real numbers;

2. *understand* the difference between sequences and series;

3. distinguish between convergence and/or divergence of some standard series or sequences;

4. apply successive approximations to roots of equations;

5. *develop* the ability to use concept to model and solve real-world problems.

SPECIFIC OBJECTIVES

(A) Sequences

Students should be able to:

1. define the concept of a sequence $\{a_n\}$ of terms a_n as a function from the positive integers to the real numbers;

2. write a specific term from the formula for the n^{th} term, or from a recurrence relation;

3. describe the behaviour of convergent and divergent sequences, through simple examples;

4. apply mathematical induction to establish properties of sequences.

CONTENT

(A) Sequences

1. Definition, convergence, divergence *and* limit of a sequence.

2. Sequences defined by recurrence relations.

3. Application of mathematical induction to sequences.

UNIT 2
MODULE 2: SEQUENCES, SERIES AND APPROXIMATIONS (cont'd)

SPECIFIC OBJECTIVES

(B) Series

Students should be able to:

1. use the summation (Σ) notation;

2. define a series, as the sum of the terms of a sequence;

3. identify the n^{th} term of a series, in the summation notation;

4. define the m^{th} partial sum S_m as the sum of the first m terms of the sequence, that is,

   $S_m = \sum_{r=1}^{m} a_r$;

5. apply mathematical induction to establish properties of series;

6. find the sum to infinity of a convergent series;

7. apply the method of differences to appropriate series, and find their sums;

8. use the Maclaurin theorem for the expansion of series;

9. *use the Taylor theorem for the expansion of series*.

CONTENT

(B) Series

1. Summation notation (Σ).

2. *Series as the sum of terms of a sequence.*

3. *Convergence and/or divergence of series to which the method of differences can be applied.*

4. *The Maclaurin series.*

5. *The Taylor series.*

6. Applications of mathematical induction to series.

SPECIFIC OBJECTIVES

(C) The Binomial Theorem

Students should be able to:

1. explain the meaning and use simple properties of $n!$ and $\binom{n}{r}$, that is, ${}^{n}C_{r}$, where $n, r \in \mathbb{Z}$;
2. *recognise that* ${}^{n}C_{r}$ *that is,* $\binom{n}{r}$, *is the number of ways in which r objects may be chosen from n distinct objects;*
3. expand $(a + b)^n$ for $n \in \mathbb{Q}$;
4. apply the Binomial Theorem to real-world problems, for example, in mathematics of finance, science.

CONTENT

(C) The Binomial Theorem

1. Factorials and Binomial coefficients; their interpretation and properties.
2. The Binomial Theorem.
3. Applications of the Binomial Theorem.

SPECIFIC OBJECTIVES

(D) Roots of Equations

Students should be able to:

1. test for the existence of a root of $\mathrm{f}(x) = 0$ where f is continuous using the Intermediate Value Theorem;
2. *use interval bisection to find an approximation for a root in a given interval;*
3. *use linear interpolation to find an approximation for a root in a given interval;*
4. explain, in geometrical terms, the working of the Newton-Raphson method;

5. use the Newton-Raphson method to find successive approximations to the roots of $f(x) = 0$, where f is differentiable;

6. *use a given iteration to determine a root of an equation to a specified degree of accuracy.*

CONTENT

(D) Roots of Equations

Finding successive approximations to roots of equations using:

1. *Intermediate Value Theorem;*

2. *Interval Bisection;*

3. *Linear Interpolation;*

4. *Newton - Raphson Method;*

5. *Iteration.*

<u>Suggested Teaching and Learning Activities</u>

To facilitate students' attainment of the objectives of this Module, teachers are advised to engage students in the learning activities listed below.

1. The Binomial Theorem

Students may be motivated to do this topic by having successive expansions of $(a + x)^n$ and then investigating the coefficients obtained when expansions are carried out.

$$(a + b)^1 = a + b$$

$$(a + b)^2 = a^2 + 2ab + b^2$$

$$(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$

$$(a + b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + a^4$$

and so on.

By extracting the coefficients of each term made up of powers of a, x or a and x.

$$\begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & 1 & & 1 & & & \\ & & 1 & & 2 & & 1 & & \\ & 1 & & 3 & & 3 & & 1 & \\ 1 & & 4 & & 6 & & 4 & & 1 \end{array}$$

Students should be encouraged to use the emerging pattern to generate further expansions of $(a + x)^n$. This can be done by generating the coefficients from Pascal's Triangle and then investigating other patterns. For example, by looking at the powers of a and x (powers of x increase from 0 to n, while powers of a decrease from n to 0; powers of a and x add up to n).

In discussing the need to find a more efficient method of doing the expansions, the Binomial Theorem may be introduced. However, this can only be done after the students are exposed to principles of counting, with particular reference to the process of <u>selecting</u>. In so doing, teachers will need to guide students through appropriate examples involving the selection of r objects, say, from a group of n unlike objects. This activity can lead to defining ${}^nC_r$ as the number of ways of selecting r objects from a group of n unlike objects.

In teaching this principle, enough examples should be presented before ${}^nC_r = \dfrac{n!}{(n-r)!\,r!}$ formula is developed.

The binomial theorem may then be established by using the expansion of $(1 + x)^n$ as a starting point. A suggested approach is given below:

Consider $(1 + x)^n$.

To expand, the student is expected to multiply $(1 + x)$ by itself n times, that is,
$(1 + x)^n = (1 + x)(1 + x)(1 + x) \ldots (1 + x)$.

The result of the expansion is found as given below:

The constant term is obtained by multiplying all the 1's. The result is therefore 1.

The term in x is obtained by multiplying $(n - 1)$ 1's and one x. This x, however, may be chosen from any of the n brackets. That is, we need to choose one x out of n different brackets. This can be done in ${}^nC_1$ ways. Hence, the coefficient of x is ${}^nC_1$.

Similarly, the term in x^2 may be obtained by choosing two x's and $(n - 2)$ 1's. The x's may be chosen from any two of the n brackets. This can be done in ${}^nC_2$ ways. The coefficient of x^2 is therefore ${}^nC_2$.

This process continues and the expansion is obtained:

$$(1 + x)^n = 1 + {}^nC_1 x + {}^nC_2 x^2 + {}^nC_3 x^3 + \ldots + x^n.$$

This is known as the binomial theorem. The theorem may be written as

$$(1 + x)^n = \sum_{r=0}^{n} {}^nC_r x^r.$$

The generalisation of this could be done as a class activity where students are asked to show that:

$$(a + b)^n = a^n + {}^nC_1 a^{n-1} b + {}^nC_2 a^{n-2} b^2 + {}^nC_3 a^{n-3} b^3 + \ldots + b^n.$$

This is the binomial expansion of $(a + b)^n$ for positive integral values of n. The expansion terminates after $(n + 1)$ terms.

2. The Intermediate Value Theorem

(a) Motivate with an example.

Example: A taxi is travelling at 5 km/h at 8:00 a.m. Fifteen minutes later the speed is 100 km/h. Since the speed varies continuously, clearly at some time between 8:00 a.m. and 8:15 a.m. the taxi was travelling at 75 km/h.

Note that the taxi could have traveled at 75 km/h at more than one time between 8:00 a.m. and 8:15 a.m.

(b) Use examples of continuous functions to illustrate the Intermediate Value Theorem.

Example: $f(x) = x^2 - x - 6$ examined on the intervals (3.5, 5) and (0, 4).

3. Existence of Roots

Introduce the existence of the root of a continuous function $f(x)$ between given values a and b as an application of the Intermediate Value Theorem.

Emphasis should be placed on the fact that:

(a) f must be continuous between a and b;

(b) The product of f (a) and f (b) is less than zero, that is, f (a) and f (b) must have opposite signs.

4. *Interval Bisection*

Initially students should be able to determine an interval in which a real root lies. If f *(a) and* f *(b) are of opposite signs, and* f *is continuous, then* $a < x < b$, *for the equation* f (x) = 0.

Students may be asked to investigate $x = \dfrac{\alpha + \beta}{2}$ *and note the resulting sign to determine which side of* $\dfrac{\alpha + \beta}{2}$ *the root lies. This method can be repeated until same answer to the desired degree of accuracy is obtained.*

5. *Linear Interpolation*

Given the points $(x_0,\ y_0)$ and $(x_1,\ y_1)$ *on a continuous curve* y = f (x), *students can establish that for* $\mathrm{f}\,(x_0)$ *and* $\mathrm{f}\,(x_1)$ *with opposite signs and that* f *is continuous, then* $x_0 < x < x_1$, *for the equation* f (x) = 0. *If* $|\mathrm{f}\,(x_0)| < |\mathrm{f}\,(x_1)|$ *say, students can be introduced to the concept of similar triangles to find successive approximations, holding* $\mathrm{f}\,(x_1)$ *constant. This intuitive approach is formalised in* ***linear interpolation****, where the two points* $(x_0,\ y_0)$ and $(x_1,\ y_1)$ *can be joined by a straight line and the x-value of the point on this line is calculated. A first approximation for x can be found using*

$$\frac{x}{\mathrm{f}\,(x_0)} = \frac{x_1}{\mathrm{f}\,(x_1)}\ .$$

Successive approximations can be found with this approach until the same answer to the desired degree of accuracy is obtained.

RESOURCE

Bostock, L. and Chandler, S. — *Core Mathematics for A-Levels,* United Kingdom*:* Stanley Thornes Publishing Limited, 1997.

Campbell, E. — *Pure Mathematics for CAPE, Vol. 2, Jamaica:* LMH Publishing Limited, 2007.

UNIT 2
MODULE 3: COUNTING, MATRICES AND *DIFFERENTIAL EQUATIONS*

GENERAL OBJECTIVES

On completion of this Module, students should:

1. *develop* the ability to analyse and solve simple problems dealing with choices and arrangements;

2. *develop an understanding of* the algebra of matrices;

3. *develop* the ability to analyse and solve systems of linear equations;

4. *develop skills to model some real-world phenomena by means of differential equations, and solve these;*

5. *develop* the ability to use concepts to model and solve real-world problems.

SPECIFIC OBJECTIVES

(A) Counting

Students should be able to:

1. state the principles of counting;

2. find the number of ways of arranging n distinct objects;

3. find the number of ways of arranging n objects some of which are identical;

4. find the number of ways of choosing r distinct objects from a set of n distinct objects;

5. identify a sample space*;*

6. *identify* the numbers of possible outcomes in a given sample space*;*

7. *use Venn diagrams to illustrate the principles of counting;*

8. *use possibility space diagram to identify a sample space*;

9. *define* and calculate P(A), the probability of an event A occurring as the number of possible ways in which A can occur divided by the total number of possible ways in which all equally likely outcomes, including A, occur;

10. use the fact that $0 \leq \mathrm{P}\,(A) \leq 1$;

11. demonstrate and use the property that the total probability for all possible outcomes in the sample space is 1;

12. use the property that P(A') = $1 - \text{P}(A)$ is the probability that event A does not occur*;*

13. use the property P($A \cup B$) = P (A) + P (B) – P($A \cap B$) for event A and B;

14. use the property P($A \cap B$) = 0 or P ($A \cup B$) = P (A) + P (B), where A and B are mutually exclusive events;

15. use the property P($A \cap B$) = P(A) × P(B), where A and B are independent events;

16. *use the property* P(A|B) = $\dfrac{\text{P}(A \cap B)}{\text{P}(B)}$ *where* P(B) $\neq$ *0.*

17. *use a tree diagram to list all possible outcomes for conditional probability.*

CONTENT

(A) Counting

1. Principles of counting.

2. Arrangements with and without repetitions.

3. Selections.

4. Venn diagram.

5. Possibility space diagram.

6. Concept of probability and elementary applications.

7. *Tree diagram*.

SPECIFIC OBJECTIVES

(B) Matrices and Systems of Linear Equations

Students should be able to:

1. operate with conformable matrices, carry out simple operations and manipulate matrices using their properties;

2. evaluate the determinants of $n \times n$ matrices, $1 \leq n \leq 3$;

3. reduce a system of linear equations to echelon form;

4. row-reduce the augmented matrix of an $n \times n$ system of linear equations, $n = 2, 3$*;*

5. determine whether the system is consistent, and if so, how many solutions it has;

6. find all solutions of a consistent system;

7. invert a non-singular 3×3 matrix;

8. solve a 3×3 system of linear equations, having a non-singular coefficient matrix, by using its inverse.

CONTENT

(B) Matrices and Systems of Linear Equations

1. $m \times n$ matrices, for $1 \leq m \leq 3$, and $1 \leq n \leq 3$, and equality of matrices.

2. Addition of conformable matrices, zero matrix and additive inverse, associativity, commutativity, distributivity, transposes.

3. Multiplication of a matrix by a scalar.

4. Multiplication of conformable matrices.

5. Square matrices, singular and non-singular matrices, unit matrix and multiplicative inverse.

6. $n \times n$ determinants, $1 \leq n \leq 3$.

7. $n \times n$ systems of linear equations, consistency of the systems, equivalence of the systems, solution by reduction to row echelon form, $n = 2, 3$.

8. $n \times n$ systems of linear equations by row reduction of an augmented matrix, $n = 2, 3$.

SPECIFIC OBJECTIVES

(C) *Differential Equations and Modeling*

Students should be able to:

1. *solve first order linear differential equations* $y' - ky = \mathrm{f}\,(x)$ *using an integrating factor, given that* k *is a real constant or a function of* x*, and* f *is a function;*

2. *solve first order linear differential equations given boundary conditions;*

3. *solve second order ordinary differential equations with constant coefficients of the form*

$$ay'' + by' + cy = 0 = \text{f}\,(x), \text{ where } a, b, c \in \mathbb{R} \text{ and f}\,(x) \text{ is:}$$

   (a) *a polynomial,*
   (b) *an exponential function,*
   (c) *a trigonometric function;*

   and the complementary function may consist of

   (a) *2 real and distinct root,*
   (b) *2 equal roots,*
   (c) *2 complex roots*;

4. *solve second order ordinary differential equation given boundary conditions;*

5. *use substitution to reduce a second order ordinary differential equation to a suitable form.*

CONTENT

(C) ***Differential Equations and Modeling***

1. *Formulation and solution of differential equations of the form* $y' - ky = \text{f}\,(x)$*, where* k *is a real constant or a function of* x*, and* f *is a function.*

2. *Second order ordinary differential equations.*

Suggested Teaching and Learning Activities

To facilitate students' attainment of the objectives of this Module, teachers are advised to engage students in the teaching and learning activities listed below.

1. Counting

Consider the three scenarios given below.

(a) Throw two dice. Find the probability that the sum of the dots on the uppermost faces of the dice is 6.

(b) An insurance salesman visits a household. What is the probability that he will be successful in selling a policy?

(c) A hurricane is situated 500km east of Barbados. What is the probability that it will hit the island?

These three scenarios are very different for the calculation of probability. In 'a', the probability is calculated as the number of successful outcomes divided by the total possible number of outcomes. In this classical approach, the probability assignments are based on equally likely outcomes and the entire sample space is known from the start.

The situation in 'b' is no longer as well determined as in 'a'. It is necessary to obtain historical data for the salesman in question and estimate the required probability by dividing the number of successful sales by the total number of households visited. This frequency approach still relies on the existence of data and its applications are more realistic than those of the classical methodology.

For 'c' it is very unclear that a probability can be assigned. Historical data is most likely unavailable or insufficient for the frequency approach. The statistician might have to revert to informed educated guesses. This is quite permissible and reflects the analyst's prior opinion. This approach lends itself to a Bayesian methodology.

One should note that the rules and results of probability theory remain exactly the same regardless of the method used to estimate the probability of events.

2. Systems of Linear Equations in Two Unknowns

(a) In order to give a geometric interpretation, students should be asked to plot on graph paper the pair of straight lines represented by a given pair of linear equations in two unknowns, and to examine the relationship between the pair of straight lines in the cases where the system of equations has been shown to have:

(i) one solution;

(ii) many solutions;

(iii) no solutions.

(b) Given a system of equations with a unique solution, there exist equivalent systems, obtained by row-reduction, having the same solution. To demonstrate this, students should be asked to plot on the same piece of graph paper all the straight lines represented by the successive pairs of linear equations which result from each of the row operations used to obtain the solution.

UNIT 2
MODULE 3: *COUNTING, MATRICES AND DIFFERENTIAL EQUATIONS (cont'd)*
RESOURCES

Bolt, B. and Hobbs, D.	*101 Mathematical Projects: A Resource Book*, United Kingdom: Cambridge University Press, 1994.
Bostock, L. and Chandler, S.	*Core Mathematics for A-Levels*, United Kingdom: Stanley Thornes Publishing Limited, 1997.
Campbell, E.	*Pure Mathematics for CAPE, Vol. 2, Jamaica:* LMH Publishing Limited, 2007.
Crawshaw, J. and Chambers, J.	*A Concise Course in A-Level Statistics*, Cheltenham, United Kingdom: Stanley Thornes (Publishers) Limited, 1999.

♦ OUTLINE OF ASSESSMENT

Each Unit of the syllabus is assessed separately. The scheme of assessment for each Unit is the same. A candidate's performance on each Unit is reported as an overall grade and a grade on each Module of the Unit. The assessment comprises two components, one external and one internal.

EXTERNAL ASSESSMENT **(80 per cent)**

The candidate is required to sit two written papers for a total of 4 hrs.

Paper 01 (1 hour 30 minutes)	This paper comprises forty-five, compulsory multiple-choice items.	**30 per cent**
Paper 02 (2 hours 30 minutes)	This paper comprises six, compulsory extended-response questions.	**50 per cent**

SCHOOL-BASED ASSESSMENT **(20 per cent)**

School-Based Assessment in respect of each Unit will contribute 20 per cent to the total assessment of a candidate's performance on that Unit.

Paper 03/1

This paper is intended for candidates registered through a school or other approved educational institution.

The School-Based Assessment comprises three class tests designed and assessed internally by the teacher and externally by CXC. The duration of each test is 1 to 1½ hours. The tests must span, individually or collectively, the three Modules, and must include mathematical modelling.

Paper 03/2 (Alternative to Paper 03/1)

This paper is an alternative to Paper 031 and is intended for private candidates.

The paper comprises three questions. The duration of the paper is 1½ hours.

MODERATION OF SCHOOL-BASED ASSESSMENT (PAPER 03/1)

School-Based Assessment Record Sheets are available online via the CXC's website www.cxc.org.

All School-Based Assessment Record of marks must be submitted online using the SBA data capture module of the Online Registration System (ORS). A sample of assignments will be requested by CXC for moderation purposes. These assignments will be re-assessed by CXC Examiners who moderate the School-Based Assessment. Teachers' marks may be adjusted as a result of moderation. The Examiners' comments will be sent to schools. All samples must be delivered to the specified marking venues by the stipulated deadlines.

Copies of the students' assignment that are not submitted must be retained by the school until three months after publication by CXC of the examination results.

ASSESSMENT DETAILS FOR EACH UNIT

External Assessment by Written Papers (80 per cent of Total Assessment)

Paper 01 (1 hour 30 minutes – 30 per cent of Total Assessment)

1. **Composition of the Paper**

   (a) This paper consists of forty-five multiple-choice items, with fifteen items based on each Module.

   (b) All items are compulsory.

2. **Syllabus Coverage**

   (a) Knowledge of the entire syllabus is required.

   (b) The paper is designed to test a candidate's knowledge across the breadth of the syllabus.

3. **Question Type**

   Questions may be presented using words, symbols, tables, diagrams or a combination of these.

4. **Mark Allocation**

   (a) Each item is allocated 1 mark.

   (b) Each Module is allocated 15 marks.

   (c) The total marks available for this paper is 45.

   (d) This paper contributes 30 per cent towards the final assessment.

5. **Award of Marks**

   Marks will be awarded for reasoning, algorithmic knowledge and conceptual knowledge.

Reasoning:	Selection of appropriate strategy, evidence of clear thinking, explanation and/or logical argument.
Algorithmic Knowledge:	Evidence of knowledge, ability to apply concepts and skills, and to analyse a problem in a logical manner.
Conceptual Knowledge:	Recall or selection of facts or principles; computational skill, numerical accuracy, and acceptable tolerance limits in drawing diagrams.

6. **Use of Calculators**

(a) Each candidate is required to have a silent, non-programmable calculator for the duration of the examination, and is entirely responsible for its functioning.

(b) The use of calculators with graphical displays will not be permitted.

(c) Answers found by using a calculator, without relevant working shown, may not be awarded full marks.

(d) Calculators must not be shared during the examination.

7. **Use of Mathematical Tables**

A booklet of mathematical formulae will be provided.

Paper 02 (2 hours 30 minutes – 50 per cent of Total Assessment)

1. **Composition of Paper**

(a) The paper consists of six questions. Two questions are based on each Module (Module 1, Module 2 and Module 3).

(b) All questions are compulsory.

2. **Syllabus Coverage**

(a) Each question may be based on one or more than one topic in the Module from which the question is taken.

(b) Each question may develop a single theme or unconnected themes.

3. **Question Type**

(a) Questions may require an extended response.

(b) Questions may be presented using words, symbols, tables, diagrams or a combination of these.

4. **Mark Allocation**

(a) Each question is worth 25 marks.

(b) The number of marks allocated to each sub-question will appear in brackets on the examination paper.

(c) Each Module is allocated 50 marks.

(d) The total marks available for this paper is 150.

(e) This paper contributes 50 per cent towards the final assessment.

5. Award of Marks

(a) Marks will be awarded for reasoning, algorithmic knowledge and conceptual knowledge.

Reasoning:	Selection of appropriate strategy, evidence of clear thinking, explanation and/or logical argument.
Algorithmic Knowledge:	Evidence of knowledge, ability to apply concepts and skills, and to analyse a problem in a logical manner.
Conceptual Knowledge:	Recall or selection of facts or principles; computational skill, numerical accuracy, and acceptable tolerance limits in drawing diagrams.

(b) Full marks will be awarded for correct answers and presence of appropriate working.

(c) Where an incorrect answer is given, credit may be awarded for correct method provided that the working is shown.

(d) If an incorrect answer in a previous question or part-question is used later in a section or a question, then marks may be awarded in the latter part even though the original answer is incorrect. In this way, a candidate is not penalised twice for the same mistake.

(e) A correct answer given with no indication of the method used (in the form of written working) will receive no marks. Candidates are, therefore, advised to show all relevant working.

6. Use of Calculators

(a) Each candidate is required to have a silent, non-programmable calculator for the duration of the examination, and is responsible for its functioning.

(b) The use of calculators with graphical displays will not be permitted.

(c) Answers found by using a calculator, without relevant working shown, may not be awarded full marks.

(d) Calculators must not be shared during the examination.

7. Use of Mathematical Tables

A booklet of mathematical formulae will be provided.

SCHOOL-BASED ASSESSMENT

School-Based Assessment is an integral part of student assessment in the course covered by this syllabus. It is intended to assist students in acquiring certain knowledge, skills, and attitudes that are associated with the subject. The activities for the School-Based Assessment are linked to the syllabus and should form part of the learning activities to enable the student to achieve the objectives of the syllabus.

During the course of study for the subject, students obtain marks for the competence they develop and demonstrate in undertaking their School-Based Assessment assignments. These marks contribute to the final marks and grades that are awarded to students for their performance in the examination.

The guidelines provided in this syllabus for selecting appropriate tasks are intended to assist teachers and students in selecting assignments that are valid for the purpose of School-Based Assessment. In order to ensure that the scores awarded by teachers are in line with the CXC standards, the Council undertakes the moderation of a sample of the School-Based Assessment assignments marked by each teacher.

School-Based Assessment provides an opportunity to individualise a part of the curriculum to meet the needs of students. It facilitates feedback to the students at various stages of their experience. This helps to build the self-confidence of students as they proceed with their studies. School-Based Assessment also facilitates the development of the critical skills and abilities emphasised by this CAPE subject and enhance the validity of the examination on which candidate performance is reported. School-Based assessment, therefore, makes a significant and unique contribution to both the development of relevant skills and the testing and rewarding of students for the development of those skills.

The Caribbean Examinations Council seeks to ensure that the School-Based Assessment scores are valid and reliable estimates of accomplishment. The guidelines provided in this syllabus are intended to assist in doing so.

Paper 03/1 (20 per cent of Total Assessment)

This paper comprises three tests. The tests, designed and assessed by the teacher, are externally moderated by CXC. The duration of each test is 1 to 1½ hours.

1. **Composition of the Tests**

   The three tests of which the School-Based Assessment is comprised must span, individually or collectively, the three Modules and include mathematical modelling. At least 30 per cent of the marks must be allocated to mathematical modelling.

2. **Question Type**

   Paper 03/2 may be used as a prototype but teachers are encouraged to be creative and original.

3. **Mark Allocation**

   (a) There is a maximum of 20 marks for each test.

   (b) There is a maximum of 60 marks for the School-Based Assessment.

(c) The candidate's mark is the total mark for the three tests. One-third of the total marks for the three tests is allocated to each of the three Modules. (See 'General Guidelines for Teachers' below.)

(d) For each test, marks should be allocated for the skills outlined on page 3 of this Syllabus.

4. Award of Marks

(a) Marks will be awarded for reasoning, algorithmic knowledge and conceptual knowledge.

For each test, the 20 marks should be awarded as follows:

Reasoning:	Selection of appropriate strategy, evidence of clear thinking, explanation and/or logical argument. (3 – 5 marks)
Algorithmic Knowledge:	Evidence of knowledge, ability to apply concepts and skills, and to analyse a problem in a logical manner. (10 – 14 marks)
Conceptual Knowledge:	Recall or selection of facts or principles; computational skill, numerical accuracy, and acceptable tolerance limits in drawing diagrams. (3 – 5 marks)

(b) If an incorrect answer in an earlier question or part-question is used later in a section or a question, then marks may be awarded in the later part even though the original answer is incorrect. In this way, a candidate is <u>not</u> penalised twice for the same mistake.

(c) A correct answer given with no indication of the method used (in the form of written working) will receive no marks. Candidates should be advised to show <u>all</u> relevant working.

<u>Paper 03/2 (20 per cent of Total Assessment)</u>

1. Composition of Paper

(a) This paper consists of three questions, each based on one of the three Modules.

(b) All questions are compulsory.

2. Question Type

(a) Each question may require an extended response.

(b) A part of or an entire question may focus on mathematical modeling.

(c) A question may be presented using words, symbols, tables, diagrams or a combination of these.

3. **Mark Allocation**

(a) Each question carries a maximum of 20 marks.

(b) The Paper carries a maximum of 60 marks.

(c) For each question, marks should be allocated for the skills outlined on page 3 of this Syllabus.

4. **Award of Marks**

(a) Marks will be awarded for reasoning, algorithmic knowledge and conceptual knowledge.

For each test, the 20 marks should be awarded as follows:

Reasoning:	Selection of appropriate strategy, evidence of clear reasoning, explanation and/or logical argument. (3 – 5 marks)
Algorithmic Knowledge:	Evidence of knowledge, ability to apply concepts and skills, and to analyse a problem in a logical manner. (10 – 14 marks)
Conceptual Knowledge:	Recall or selection of facts or principles; computational skill, numerical accuracy, and acceptable tolerance limits in drawing diagrams. (3 – 5 marks)

(b) If an incorrect answer in a previous question or part-question is used later in a section or a question, then marks may be awarded in the later part even though the original answer is incorrect. In this way, a candidate is not penalised twice for the same mistake.

(c) A correct answer given with no indication of the method used (in the form of written working) will receive no marks. Candidates should be advised to show all relevant working.

GENERAL GUIDELINES FOR TEACHERS

1. Teachers should note that the reliability of marks awarded is a significant factor in the *School-Based Assessment*, and has far-reaching implications for the candidate's final grade.

2. Candidates who do not fulfill the requirements of the *School-Based Assessment* will be considered absent from the whole examination.

3. Teachers are asked to note the following:

(a) the relationship between the marks for the assignment and those submitted to CXC on the *school-based assessment* form should be clearly shown;

(b) the teacher is required to allocate one-third of the total score for the *School-Based Assessment* to each Module. Fractional marks should not be awarded. In cases where the mark is not divisible by three, then:

(i) when the remainder is 1 mark, the mark should be allocated to Module 3;

(ii) when the remainder is 2, then a mark should be allocated to Module 3 and the other mark to Module 2;

for example, 35 marks would be allocated as follows:

35/3 = 11 remainder 2 so 11 marks to Module 1 and 12 marks to each of Modules 2 and 3.

(c) the standard of marking should be consistent.

4. Teachers are required to submit a copy of EACH test, the solutions and the mark schemes with the sample.

♦ REGULATIONS FOR PRIVATE CANDIDATES

Candidates who are registered privately will be required to sit Paper 01, Paper 02 and Paper 03/2. Paper 03/2 will be 1½ hours' duration and will consist of three questions, each worth 20 marks. Each question will be based on the objectives and content of one of the three Modules of the Unit. Paper 03/2 will contribute 20 per cent of the total assessment of a candidate's performance on that Unit.

Paper 03/2 (1 hour 30 minutes)

The paper consists of three questions. Each question is based on the topics contained in one Module and tests candidates' skills and abilities to:

1. recall, select and use appropriate facts, concepts and principles in a variety of contexts;
2. manipulate mathematical expressions and procedures using appropriate symbols and language, logical deduction and inferences;
3. select and use a simple mathematical model to describe a real-world situation;
4. simplify and solve mathematical models;
5. interpret mathematical results and their application in a real-world problem.

♦ REGULATIONS FOR RE-SIT CANDIDATES

Candidates who have earned a moderated score of at least 50 per cent of the total marks for the Internal Assessment component, may elect not to repeat this component, provided they re-write the examination no later than TWO years following their first attempt. These resit candidates must complete Papers 01 and 02 of the examination for the year in which they register.

Resit candidates must be entered through a school or other approved educational institution.

Candidates who have obtained less than 50 per cent of the marks for the School-Based Assessment component must repeat the component at any subsequent sitting or write Paper 03/2.

♦ ASSESSMENT GRID

The Assessment Grid for each Unit contains marks assigned to papers and to Modules and percentage contributions of each paper to total scores.

Units 1 and 2

Papers	Module 1	Module 2	Module 3	Total	(%)
External Assessment Paper 01 (1 hour 30 minutes)	15 (30 weighted)	15 (30 weighted)	15 (30 weighted)	45 (90 weighted)	(30)
Paper 02 (2 hours 30 minutes)	50	50	50	150	(50)
School-Based Assessment Paper 03/1 or Paper 03/2 (1 hour 30 minutes)	20	20	20	60	(20)
Total	100	100	100	300	(100)

♦ MATHEMATICAL NOTATION

The following list summarises the notation used in the Mathematics papers of the Caribbean Advanced Proficiency Examination.

Set Notation

$\in$	is an element of
$\notin$	is not an element of
$\{x: \ldots\}$	the set of all x such that $\ldots$
$n(\mathrm{A})$	the number of elements in set A
$\varnothing$	the empty set
U	the universal set
A'	the complement of the set A
$\mathbb{W}$	the set of whole numbers {0, 1, 2, 3, $\ldots$}
$\mathbb{N}$	the set of natural numbers {1, 2, 3, $\ldots$}
$\mathbb{Z}$	the set of integers
$\mathbb{Q}$	the set of rational numbers
$\overline{\mathbb{Q}}$	*the set of irrational numbers*
$\mathbb{R}$	the set of real numbers
$\mathbb{C}$	the set of complex numbers
$\subset$	is a *proper* subset of
$\not\subset$	is not a proper subset of
$\subseteq$	*is a subset of*
$\nsubseteq$	is not a subset of
$\cup$	union
$\cap$	intersection
$[a, b]$	the closed interval $\{x \in \mathbb{R}: a \leq x \leq b\}$
(a, b)	the open interval $\{x \in \mathbb{R}: a < x < b\}$
$[a, b)$	the interval $\{x \in \mathbb{R}: a \leq x < b\}$
$(a, b]$	the interval $\{x \in \mathbb{R}: a < x \leq b\}$

Logic

$\wedge$	conjunction
$\vee$	(inclusive) injunction
$\veebar$	exclusive disjunction
$\sim$	negation
$\rightarrow$	conditionality
$\leftrightarrow$	bi-conditionality
$\Rightarrow$	implication
$\Leftrightarrow$	equivalence

Miscellaneous Symbols

$\equiv$	is identical to
$\approx$	is approximately equal to
$\propto$	is proportional to
∞	infinity

Operations

$\sum_{i=1}^{n} x_i$	$x_1 + x_2 + \ldots + x_n$
$\sqrt{x}$	the positive square root of the real number x
$\lvert x \rvert$	the modulus of the real number x
$n!$	n factorial, $1 \times 2 \times \ldots \times n$ for $n \in \mathrm{N}$ $(0! = 1)$
${}^nC_r \binom{n}{r}$	the binomial coefficient, $\dfrac{n!}{(n-r)!\,r!}$, for $n, r \in \mathbb{N}, 0 \le r \le n$
${}^nP_r$	$\dfrac{n!}{(n-r)!}$

Functions

f	the function f
f(x)	the value of the function f at x
f: $A \to B$	the function f under which each element of the set A has an image in the set B
f: $x \to y$	the function f maps the element x to the element y
f^{-1}	the inverse of the function f
fg	the composite function f(g(x))
$\lim_{x \to a} \mathrm{f}(x)$	the limit of f(x) as x tends to a
Δx, δx	an increment of x
$\dfrac{dy}{dx}$, y'	the *first* derivative of y with respect to x
$\dfrac{d^ny}{dx^n}$, $y^{(n)}$	the nth derivative of y with respect to x
$\mathrm{f}'(x), \mathrm{f}''(x), \cdots, \mathrm{f}^{(n)}(x)$	the first, second, …, nth derivatives of f(x) with respect to x
$\dot{x}, \ddot{x}$	the first and second derivatives of x with respect to time t
e	the exponential constant
ln x	the natural logarithm of x (to base e)
lg x	the logarithm of x to base 10

Complex Numbers

i	$\sqrt{-1}$
z	a complex number, $z = x + y\mathrm{i}$ where $x, y \in \mathrm{P}$
Re z	the real part of z
Im z	the imaginary part of z
$\lvert z \rvert$	the modulus of z
arg z	the argument of z, where $-\pi < \arg z \le \pi$
$\bar{z}$, z^*	the complex conjugate of z

Vectors

a, **a**, $\overrightarrow{AB}$	vectors
$\hat{\mathbf{a}}$	a unit vector in the direction of the vector **a**
$\mid \mathbf{a} \mid$	the magnitude of the vector **a**
a.**b**	the scalar product of the vectors **a** and **b**
i, **j**, **k**	unit vectors in the directions of the positive Cartesian coordinate axes
$\begin{pmatrix} x \\ y \\ z \end{pmatrix}$	$x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$

Probability

S	the sample space
$A, B, \ldots$	the events $A, B, \ldots$
$\mathrm{P}\ (A')$	the probability that the event A *does not occur*

Matrices

M	a matrix M
(M^{-1})	inverse of the non-singular square matrix M
M^{T}, M_{T}	transpose of the matrix M
det M, $\mid \mathrm{M} \mid$	determinant of the square matrix M

Western Zone Office

CARIBBEAN EXAMINATIONS COUNCIL

Caribbean Advanced Proficiency Examination®

PURE MATHEMATICS

Specimen Papers and Mark Schemes/Keys

Specimen Papers:	-	Unit 1, Paper 01 Unit 1, Paper 02 Unit 1, Paper 032 Unit 2, Paper 01 Unit 2, Paper 02 Unit 2, Paper 032
Mark Schemes and Keys:	-	Unit 1, Paper 01 Unit 1, Paper 02 Unit 1, Paper 032 Unit 2, Paper 01 Unit 2, Paper 02 Unit 2, Paper 032

FORM TP 02134010/SPEC

C A R I B B E A N   E X A M I N A T I O N S   C O U N C I L

ADVANCED PROFICIENCY EXAMINATION

SPECIMEN PAPER
MULTIPLE CHOICE QUESTIONS
FOR

PURE MATHEMATICS

UNIT 1 – Paper 01

ALGEBRA, GEOMETRY AND CALCULUS

90 minutes

READ THE FOLLOWING INSTRUCTIONS CAREFULLY.

1. This test consists of 45 items. You will have 90 minutes to answer them.

2. Each item in this test has four suggested answers lettered (A), (B), (C), (D). Read each item you are about to answer and decide which choice is best.

3. Look at the sample item below.

   Sample Item

   The lines $2y - 3x - 13 = 0$ and $y + x + 1 = 0$ intersect at the point

   (A) (–3, –2)
   (B) (–3, 2)
   (C) (3, –2)
   (D) (3, 2)

   Sample Answer

   Ⓐ ● Ⓒ Ⓓ

   The best answer to this item is "(–3, 2)", so answer space (B) has been shaded.

4. You may do any rough work in this booklet.

5. The use of silent, non-programmable scientific calculators is allowed.

Examination Materials Permitted

A list of mathematical formulae and tables (provided) – **Revised 2010**

DO NOT TURN THIS PAGE UNTIL YOU ARE TOLD TO DO SO

1. Let **p**, **q** and **r** be the propositions
p: Students have a driving licence,
q: Students have a passport,
r: Students have an identification card.

The compound proposition, Students have a driving licence or identification card (but not both) together with a passport is expressed as

(A) $((\mathbf{p} \wedge \mathbf{r}) \vee \sim(\mathbf{p} \wedge \mathbf{r})) \wedge \mathbf{q}$
(B) $((\mathbf{p} \vee \mathbf{r}) \vee \sim(\mathbf{p} \wedge \mathbf{r})) \vee \mathbf{q}$
(C) $((\mathbf{p} \vee \mathbf{r}) \vee \sim(\mathbf{p} \vee \mathbf{r})) \wedge \mathbf{q}$
(D) $((\mathbf{p} \vee \mathbf{r}) \wedge \sim(\mathbf{p} \wedge \mathbf{r})) \wedge \mathbf{q}$

2. The compound proposition $\mathbf{p} \wedge \mathbf{q}$ is true can be illustrated by the truth table

(A)

p	q	p ∧ q
0	0	0
0	1	0
1	0	0
1	1	1

(B)

p	q	p ∧ q
0	0	1
0	1	0
1	0	0
1	1	1

(C)

p	q	p ∧ q
0	0	0
0	1	0
1	0	1
1	1	1

(D)

p	q	p ∧ q
0	0	1
0	1	0
1	0	1
1	1	0

3. The contrapositive for the conditional proposition $\mathbf{p} \rightarrow \mathbf{q}$ is

(A) $\mathbf{q} \rightarrow \mathbf{p}$
(B) $\sim \mathbf{p} \rightarrow \mathbf{q}$
(C) $\sim \mathbf{q} \rightarrow \sim \mathbf{p}$
(D) $\mathbf{p} \rightarrow \sim \mathbf{q}$

4. The proposition $\mathbf{q} \rightarrow \mathbf{p}$ is logically equivalent to

(A) $\sim \mathbf{p} \wedge \sim \mathbf{q}$
(B) $\mathbf{p} \vee \sim \mathbf{q}$
(C) $\sim \mathbf{q} \wedge \mathbf{p}$
(D) $\mathbf{q} \wedge \sim \mathbf{p}$

5. The expression $\dfrac{5\sqrt{45} - \sqrt{80}}{\sqrt{5} - \sqrt{125}}$ is equal to

(A) $\dfrac{5\sqrt{35}}{\sqrt{120}}$

(B) $\dfrac{11\sqrt{5}}{5}$

(C) $\dfrac{11}{4\sqrt{5}}$

(D) $-\dfrac{11}{4}$

6. If $2x^3 + ax^2 - 5x - 1$ leaves a remainder of 3 when divided by $(2x + 1)$, then a is

(A) -7

(B) 7

(C) $-\frac{1}{2}$

(D) -5
(E)

GO ON TO THE NEXT PAGE

7. Given that $x = 3^y$, $y > 0$ then $\log_x 3$ is equal to

(A) y

(B) $3y$

(C) $\frac{1}{y}$

(D) $\frac{3}{y}$

8. Given that f $(x) = 2 - e^{2x}$, the inverse function, $f^{-1}(x)$, for $x < 2$ is

(A) $\ln(2 - x)$
(B) $\ln(2 - 2x)$
(C) $2\ln(2 - x)$
(D) $\frac{1}{2}\ln(2 - x)$

9. The function f $(x) = 2x^2 - 4x + 5$, for $x \in \mathbb{R}$, is one-to-one for $x > k$, where $k \in \mathbb{R}$. The value of k is

(A) -2
(B) -1
(C) 1
(D) 3

10. Given that fg $(x) = x$, where g $(x) = \frac{2x+1}{3}$, f $(x) =$

(A) $\frac{3x-1}{2}$
(B) $\frac{3}{2x+1}$
(C) $\frac{2}{3x+1}$
(D) $\frac{3x}{2x+1}$

11. The values of x for which $|2x - 3| = x + 1$ are

(A) $x = -\frac{2}{3}$, $x = -4$

(B) $x = -\frac{2}{3}$, $x = 4$

(C) $x = \frac{2}{3}$, $x = 4$

(D) $x = \frac{2}{3}$, $x = -4$

12. Given that $x^2 + 4x + 3$ is a factor of f $(x) = 2x^3 + 7x^2 + 2x - 3$, the values of x for which f $(x) = 0$ are

(A) $-3, 1, -\frac{1}{2}$
(B) $-3, -1, -\frac{1}{2}$
(C) $3, 1, \frac{1}{2}$
(D) $-3, -1, \frac{1}{2}$

13. The cubic equation $2x^3 + x^2 - 22x + 24 = 0$ has roots α, β and γ. The value of $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}$ is

(A) $-\frac{1}{12}$

(B) $-\frac{1}{11}$

(C) $-\frac{1}{2}$

(D) $\frac{11}{12}$

14. The range(s) of values of x for which $\frac{3x+2}{x-1} > 0$ are

(A) $x > -\frac{2}{3}$, $x > 1$
(B) $-\frac{2}{3} < x < 1$
(C) $x < \frac{2}{3}$, $x > 1$
(D) $x < -\frac{2}{3}$, $x > 1$

15. The values of x for which $|x+5| > 3$ are

(A) $x < -8, x < -2$
(B) $x > 0, x < 1$
(C) $x > -2, \quad x < -8$
(D) $x > -2, x > -8$

16. $\dfrac{1+\cot^2\theta}{\sec\theta\,\mathrm{cosec}\,\theta} \equiv$

(A) $\tan\theta$
(B) $\cos\theta$
(C) $\cot\theta$
(D) $\mathrm{cosec}\,\theta$

17. The general solution of the equation $\cos 2\theta = 1$ is

(A) $n\pi + \dfrac{\pi}{4}$

(B) $n\pi$

(C) $n\pi + \dfrac{\pi}{2}$

(D) $\dfrac{(2n+1)\pi}{4}$

18. $\cos\theta + 3\sin\theta = 2$ can be expressed as

(A) $4\cos\left(\theta - \tan^{-1}\left(\frac{1}{3}\right)\right) = 2$
(B) $2\cos\left(\theta + \tan^{-1}(3)\right) = 2$
(C) $\sqrt{10}\cos\left(\theta - \tan^{-1}(3)\right) = 2$
(D) $\sqrt{10}\cos\left(\theta + \tan^{-1}\left(\frac{1}{3}\right)\right) = 2$

19. If $\cos A = \dfrac{3}{5}$ and A is acute, then $\sin 2A$ is equal to

(A) $\dfrac{6}{25}$

(B) $\dfrac{8}{25}$

(C) $\dfrac{12}{25}$

(D) $\dfrac{24}{25}$

20. The minimum value of $\dfrac{1}{2\cos\left(\theta + \frac{\pi}{4}\right)}$ is

(A) -1

(B) 0

(C) $\frac{1}{2}$

(D) 2

21. A curve C_1 is given by the equation $y = x^2 + 1$, and a curve C_2 is given by the equation $\frac{16}{x^2} + 1$, $x \in \mathbb{R}$, $x > 0$. The value of x for which $C_1 = C_2$ is

(A) -4
(B) 2
(C) -2
(D) 4

22. The tangent to the circle, C, with equation $x^2 + y^2 + 4x - 10y - 5 = 0$ at the point $P(3, 2)$ has equation

(A) $3x + 5y - 19 = 0$
(B) $5x + 3y + 19 = 0$
(C) $3x - 5y + 19 = 0$
(D) $5x - 3y - 9 = 0$

23. The x-coordinates of the points where the line, l, with equation $y = 2x + 1$ cuts the curve C with equation $\frac{6}{x}$ are

(A) $\frac{3}{2}$, -2

(B) $-\frac{3}{2}, -2$

(C) $\frac{3}{2}$, 2

(D) $-\frac{3}{2}$, 2

24. The Cartesian equation of the curve C given by the parametric equations
$x = 3\sin\theta - 2,\ y = 4\cos\theta + 3$ is

(A) $x^2 + y^2 = 30\ \ 9$
(B) $9x^2 + 16y^2 = 13$
(C) $(x-3)^2 + 4(y-4)^2 = 36$
(D) $16(x+2)^2 + 9(y-3)^2 = 144$

25. Relative to a fixed origin, O, the position vector of A is $\overrightarrow{\mathbf{OA}} = 3\mathbf{i} - \mathbf{j} + 2\mathbf{k}$ and the position vector of B is $\overrightarrow{\mathbf{OB}} = 9\mathbf{i} - 3\mathbf{j} + 5\mathbf{k}$. $|\overrightarrow{\mathbf{AB}}|$ is

(A) 1 unit
(B) 7 units
(C) $3\sqrt{21}$ units
(D) 49 units

26. Relative to a fixed origin, O, the point A has position vector $(2\mathbf{i} + 3\mathbf{j} - 4\mathbf{k})$, and point B has position vector $(-5\mathbf{i} + 9\mathbf{j} - 5\mathbf{k})$.The line, l, passes through the points A and B. A vector equation for the line l is given by

(A) $\mathbf{r} = 2\mathbf{i} + 3\mathbf{j} - 4\mathbf{k} + \lambda(-7\mathbf{i} + 6\mathbf{j} - \mathbf{k})$
(B) $\mathbf{r} = 2\mathbf{i} + 3\mathbf{j} - 4\mathbf{k} + \lambda(-5\mathbf{i} + 9\mathbf{j} - 5\mathbf{k})$
(C) $\mathbf{r} = 2\mathbf{i} + 3\mathbf{j} - 4\mathbf{k} + \lambda(-3\mathbf{i} + 12\mathbf{j} - 9\mathbf{k})$
(D) $\mathbf{r} = 2\mathbf{i} + 3\mathbf{j} - 4\mathbf{k} + \lambda(-10\mathbf{i} + 27\mathbf{j} + 20\mathbf{k})$

27. Relative to a fixed origin, O, the line, l_1, has position vector $\begin{pmatrix} 8 \\ 13 \\ -2 \end{pmatrix} + \lambda\begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}$, and the line, l_2, has position vector $\begin{pmatrix} 8 \\ 13 \\ -2 \end{pmatrix} + \mu\begin{pmatrix} 1 \\ -4 \\ 8 \end{pmatrix}$, where λ and μ are scalars.

The cosine of the acute angle between l_1 and l_2 is given by

(A) $\cos\theta = -\frac{2}{3}$
(B) $\cos\theta = \frac{2}{3}$
(C) $\cos\theta = \frac{8}{27}$
(D) $\cos\theta = \frac{2}{27}$

28. Relative to a fixed origin, O, the point A has position vector $(10\mathbf{i} + 14\mathbf{j} - 4\mathbf{k})$, and the point B has position vector $(5\mathbf{i} + 9\mathbf{j} + 6\mathbf{k})$. Given that a vector $\mathbf{v}$ is of magnitude $3\sqrt{6}$ units in the direction of $|\overrightarrow{\mathbf{AB}}|$, then $\mathbf{v}$ =

(A) $3\mathbf{i} + 3\mathbf{j} - 6\mathbf{k}$
(B) $-3\mathbf{i} - 3\mathbf{j} + 6\mathbf{k}$
(C) $-3\mathbf{i} + 3\mathbf{j} + 6\mathbf{k}$
(D) $3\mathbf{i} - 3\mathbf{j} + 6\mathbf{k}$

29. The point P (3, 0, 1) lies in the plane π with equation $\mathbf{r} \cdot (\mathbf{i} + 4\mathbf{j} + 2\mathbf{k}) = d$. The constant d is

(A) $\sqrt{21}$

(B) 5

(C) $\sqrt{41}$

(D) 10

GO ON TO THE NEXT PAGE

30. The line, l_1, has equation $\mathbf{r} = 6\mathbf{i} - 3\mathbf{j} - 2\mathbf{k} + \lambda\,(-\mathbf{i} + 2\mathbf{j} + 4\mathbf{k})$ and the line, l_2, has equation $-5\mathbf{i} + 15\mathbf{j} + 3\mathbf{k} + \mu\,(2\mathbf{i} - 3\mathbf{j} + a\mathbf{k})$. Given that l_1 is perpendicular to l_2 the value of a is

(A) 2
(B) –2
(C) –4
(D) 6

31. $\lim\limits_{x \to 3} \dfrac{2x^2 - 5x - 3}{x^2 - 2x - 3}$ is

(A) 0
(B) 1
(C) $\frac{7}{4}$
(D) ∞

32. $\lim\limits_{\theta \to 0} \dfrac{\sin x}{\frac{x}{2}}$ is

(A) 0
(B) $\frac{1}{2}$
(C) 1
(D) 2

33. Given that $\lim\limits_{x \to -1} \{3\mathrm{f}(x) + 2\} = 11$, where f (x) is real and continuous, the $\lim\limits_{x \to -1} \{2\mathrm{f}(x) + 5x\}$ is

(A) –11
(B) 1
(C) 4
(D) 13

34. Given that f $(x) = (6x + 4)\sin x$, then $\mathrm{f}'(x)$ is

(A) $6 \cos x$
(B) $2(3x + 2)\cos x + 6\sin x$
(C) $6x \cos x + 6 \sin x$
(D) $3 + 2 \sin x + (3x + 2) \cos x$

35. The derivative by first principles of the function f $(x) = \frac{1}{x^2}$ is given by

(A) $\lim_{h \to 0} \dfrac{\frac{1}{(x+h)^2} - \frac{1}{x^2}}{h}$

(B) $\lim_{h \to 0} \dfrac{\frac{1}{(x^2+h)} - \frac{1}{x^2}}{h}$

(C) $\lim_{h \to 0} \dfrac{\frac{1}{x^2} - \frac{1}{(x^2+h)}}{h}$

(D) $\lim_{h \to 0} \frac{1}{(x+h)^2} - \frac{1}{x^2}$

36. Given f $(x) = 3 \cos 2x$, then $\mathrm{f}'(x) =$

(A) $6 \sin 2x$
(B) $-3 \sin 2x$
(C) $-6 \sin 2x$
(D) $-\frac{3}{2} \sin 2x$

37. The curve, C, with equation $y = x^3 - 6x^2 + 9x$ has stationary points at $P\,(3, 0)$ and Q $(1, 4)$. The nature of these stationary points is

(A) $(3,\ 0)_{\max}$ $(1,\ 4)_{\min}$
(B) $(3,\ 0)_{\min}$ $(1,\ 4)_{\max}$
(C) $(3,\ 0)_{\text{infl}}$ $(1,\ 4)_{\max}$
(D) $(3,\ 0)_{\text{infl}}$ $(1,\ 4)_{\min}$

38. Given that the gradient function to a curve, C, at the point $P\,(2, 3)$ is $6x^2 - 14x$, the equation of the normal to C at P is given by the equation

(A) $y - 3 = -2(x - 2)$
(B) $y - 3 = -4(x - 2)$
(C) $y - 3 = 4(x - 2)$
(D) $y - 3 = \frac{1}{4}(x - 2)$

39. $\int \frac{(2x+1)^2}{\sqrt{x}}\,\mathrm{d}x =$

(A) $\int \left(4x^{\frac{3}{2}} + 4x^{\frac{1}{2}} + x^{-\frac{1}{2}}\right)\mathrm{d}x$

(B) $\int \left(4x^{\frac{3}{2}} + x^{-\frac{1}{2}}\right)\mathrm{d}x$

(C) $\int \left(4x + 4 + x^{-1}\right)\mathrm{d}x$

(D) $\int \left(4x^{\frac{5}{2}} + 4x^{\frac{3}{2}} + x^{\frac{1}{2}}\right)\mathrm{d}x$

40. Given that $\int_1^3 \mathrm{f}(x)\,\mathrm{d}x = 8$, then

$\int_1^3 [2\mathrm{f}(x) - 5]\,\mathrm{d}x =$

(A) 6
(B) 11
(C) 13
(D) 21

41. The area of the finite region, R, enclosed by the curve $y = x - \frac{1}{\sqrt{x}}$, the lines $x = 1$ and $x = 4$ is

(A) $\frac{5}{2}$

(B) $\frac{11}{2}$

(C) $\frac{27}{4}$

(D) $\frac{19}{2}$

42. The region, R, enclosed by the curve with equation $y = 4 - x^2$ in the first quadrant is rotated completely about the ***y***-axis. The volume of the solid generated is given by

(A) $\pi\int_0^4 (4-y)\,\mathrm{d}y$

(B) $\pi\int_0^4 (4-y)^2\,\mathrm{d}y$

(C) $\pi\int_0^2 (4-y)\,\mathrm{d}y$

(D) $\pi\int_0^2 (4-y)^2\,\mathrm{d}y$

43. Given that $\frac{\mathrm{d}}{\mathrm{d}x}\frac{2x-1}{3x+2} = \frac{7}{(3x+2)^2}$ then

$\int_1^2 \frac{21}{(3x+2)^2}\,\mathrm{d}x =$

(A) $\left.\frac{3(7)}{(3x+2)^2}\right|_1^2$

(B) $\left.\frac{2x-1}{(3x+2)}\right|_1^2$

(C) $\left.\frac{3(2x-1)}{(3x+2)}\right|_1^2$

(D) $\left.\frac{-21}{(3x+2)}\right|_1^2$

44. Given that $\int_{-2}^{0} \mathrm{f}(x)\,\mathrm{d}x = \frac{16}{3}$ and water is pumped into a large tank at a rate that is proportional to its volume, V, $\int_{-2}^{2} \mathrm{f}(x)\,\mathrm{d}x = \frac{32}{3}$ where f (x) is a real continuous function in the closed interval $[-2, 2]$, then $\int_{0}^{2} \mathrm{f}(x)\,\mathrm{d}x =$

(A) $\frac{16}{3}$
(B) 16
(C) $\frac{64}{3}$
(D) 32

45. Water is pumped into a large tank at a rate that is proportional to its volume, V, at time, t, seconds. There is a small hole at the bottom of the tank and water leaks out at a constant rate of 5 m^3/s. Given that k is a positive constant, a differential equation that satisfies this situation is

(A) $\frac{\mathrm{d}V}{\mathrm{d}t} = kV - 5$

(B) $\frac{\mathrm{d}V}{\mathrm{d}t} = -kV - 5$

(C) $\frac{\mathrm{d}V}{\mathrm{d}t} = kV + 5$

(D) $\frac{\mathrm{d}V}{\mathrm{d}t} = kV$

END OF TEST

IF YOU FINISH BEFORE TIME IS CALLED, CHECK YOUR WORK ON THIS TEST.

C A R I B B E A N   E X A M I N A T I O N S   C O U N C I L

CARIBBEAN ADVANCED PROFICIENCY EXAMINATION

PURE MATHEMATICS

UNIT 1, PAPER 01

MARK SCHEME

Key

Unit 1 Paper 01

Module	Item	Key	Module	Item	Key
1	1	D	3	31	C
	2	A		32	D
	3	C		33	B
	4	B		34	B
	5	D		35	A
	6	D		36	C
	7	C		37	B
	8	D		38	D
	9	C		39	A
	10	A		40	A
	11	A		41	B
	12	B		42	A
	13	D		43	C
	14	D		44	A
	15	C		45	A
2	16	C			
	17	B			
	18	D			
	19	C			
	20	C			
	21	C			
	22	D			
	23	A			
	24	D			
	25	B			
	26	A			
	27	B			
	28	B			
	29	B			
	30	C			

FORM TP 02134020/SPEC

C A R I B B E A N   E X A M I N A T I O N S   C O U N C I L

ADVANCED PROFICIENCY EXAMINATION

PURE MATHEMATICS

UNIT 1 – Paper 02

ALGEBRA, GEOMETRY AND CALCULUS

SPECIMEN PAPER

2 hours 30 minutes

The examination paper consists of **THREE** sections: Module 1, Module 2 and Module 3.

Each section consists of 2 questions.
The maximum mark for each Module is 50.
The maximum mark for this examination is 150.
This examination consists of 8 printed pages.

READ THE FOLLOWING INSTRUCTIONS CAREFULLY.

1. **DO NOT** open this examination paper until instructed to do so.
2. Answer **ALL** questions from the **THREE** sections.
3. Write your solutions, with full working, in the answer booklet provided.
4. Unless otherwise stated in the question, any numerical answer that is not exact **MUST** be written correct to **THREE** significant figures.

Examination Materials

Graph paper (provided)
Mathematical formulae and tables (provided) – **Revised 2010**
Mathematical instruments
Silent, non- programmable, electronic calculator

DO NOT TURN THIS PAGE UNTIL YOU ARE TOLD TO DO SO

SECTION A (MODULE 1)

Answer BOTH questions.

1. (a) Let **p** and **q** be given propositions.

(i) Copy and complete the table below to show the **truth tables** of $\mathbf{p} \rightarrow \mathbf{q}$ and $\sim \mathbf{p} \vee \mathbf{q}$. **[3 marks]**

p	q	~p	p → q	~p ∨ q

(ii) Hence, state whether the compound propositions $\mathbf{p} \rightarrow \mathbf{q}$ and $\sim\mathbf{p} \vee \mathbf{q}$ are logically equivalent, giving reasons for your answer. **[2 marks]**

(iii) Use the **algebra of propositions** to show that $\mathbf{p} \wedge (\mathbf{p} \rightarrow \mathbf{q}) = \mathbf{p} \wedge \mathbf{q}$. **[3 marks]**

(b) The binary operation * is defined on the set of real numbers, $\mathbb{R}$, as follows:

$x * y = x + y - 1$ for all x, y in $\mathbb{R}$.

Prove that *

(i) is closed in $\mathbb{R}$, **[3 marks]**

(ii) is commutative in $\mathbb{R}$, **[2 marks]**

(iii) is associative in $\mathbb{R}$. **[4 marks]**

(c) Let $y = \frac{2x}{x^2 + 4}$.

(i) Show that for all real values of x, $-\frac{1}{2} \leq y \leq \frac{1}{2}$. **[5 marks]**

(ii) Hence, sketch the graph of y for all x such that $-2 \leq x \leq 2$. **[3 marks]**

Total 25 marks

GO ON TO THE NEXT PAGE

2. (a) Two of the roots of the cubic equation $2x^3 + px^2 + qx + 2 = 0$ are -1 and $\frac{1}{2}$.

Find

(i) the values of the constants p and q **[4 marks]**

(ii) the third root of the equation. **[3 marks]**

(b) Prove by Mathematical Induction that $\sum_{r=1}^{n}(6r+5) = n(3n+8)$. **[10 marks]**

(c) Solve for x the following equation $e^{2x} + 2e^{-2x} = 3$. **[8 marks]**

Total 25 marks

SECTION B (MODULE 2)

Answer BOTH questions.

3. (a) (i) Prove the identity

$$\frac{\sin 3\theta + \sin\theta}{\cos 3\theta + \cos\theta} \equiv \tan 2\theta\,. \qquad \textbf{[4 marks]}$$

(ii) Solve the equation

$$\sin 3\theta + \sin\theta + \sin 2\theta = 0,\ 0 \le \theta \le 2\pi. \qquad \textbf{[7 marks]}$$

(b) (i) Express $f(\theta) = 8\cos\theta + 6\sin\theta$ in the form $r\cos(\theta - \alpha)$

where $r > 0,\ 0^0 < \alpha < 90^0$. **[3 marks]**

(ii) Determine the minimum value of

$$g(\theta) = \frac{10}{10 + 8\cos\theta + 6\sin\theta}$$

stating the value of θ for which $g(\theta)$ is a minimum. **[4 marks]**

(c) Let A = (2, 0, 0), B = (0, 0, 2) and C = (0, 2, 0).

(i) Express the vectors $\overrightarrow{\mathbf{BC}}$ and $\overrightarrow{\mathbf{BA}}$ in the form $x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ **[2 marks]**

(ii) Show that the vector $\mathbf{r} = \mathbf{i} + \mathbf{j} + \mathbf{k}$ is perpendicular to the plane through A, B, and C. **[2 marks]**

(iii) Hence, find the Cartesian equation of the plane through A, B and C. **[3 marks]**

Total 25 marks

4. The equation of the line L is $x + 2y = 7$ and the equation of the circle C is $x^2 + y^2 - 4x - 1 = 0$.

(a) Show that the line, L, is a tangent to the circle C. **[8 marks]**

(b) Find,

(i) the equation of the tangent, M, diametrically opposite to the tangent, L, of circle C.

(ii) the equation of the diameter of C which is parallel to L **[4 marks]**

(iii) the coordinates of its points of intersection with C. **[2 marks]**

(c) The parametric equations of a curve, C, are given by

$$x = \frac{t}{1+t} \text{ and } y = \frac{t^2}{1+t}.$$

Determine the Cartesian equation of C in the form $y = f(x)$. **[6 marks]**

Total 25 marks

SECTION C (MODULE 3)

Answer BOTH questions.

5. (a) Show that $\lim_{h \to 0} \dfrac{h}{\sqrt{x+h} - \sqrt{x}} = 2\sqrt{x}$. **[6 marks]**

(b) The function f(x) is such that $f(x) = 18x + 4$. Given that $f(2) = 14$ and $f(3) = 74$, find the value of $f(4)$. **[8 marks]**

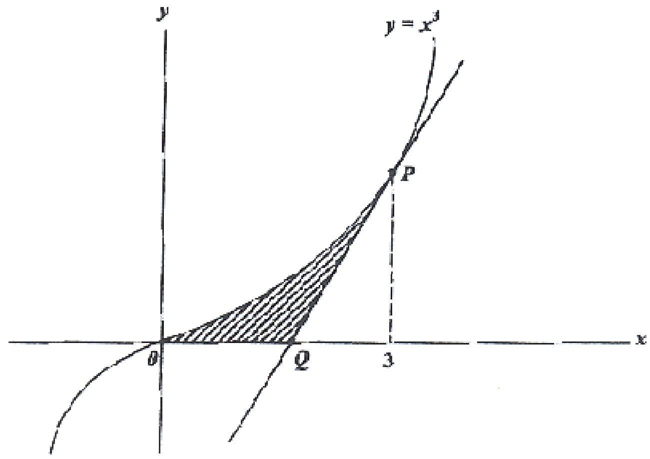
(c) If $y = \dfrac{x}{1+x^2}$, show that

(i) $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{1}{(1+x^2)^2} - y^2$ **[5 marks]**

(ii) $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} = \dfrac{2y\left(x^2-3\right)}{\left(1+x^2\right)^2}$ **[6 marks]**

Total 25 marks

6. (a) The diagram below, **not drawn to scale** is a sketch of the curve $y = x^3$. The tangent, PQ, meets the curve at $P(3, 27)$.



(i) Find

a) the equation of the tangent PQ **[4 marks]**

b) the coordinates of Q. **[1 mark]**

(ii) Calculate

a) the area of the shaded region, **[5 marks]**

b) the volume of the solid generated when the shaded region is rotated completely about the x-axis, giving your answer in terms of π. **[5 marks]**

[The volume, V, of a cone of radius r and height h is given by $V = \frac{1}{3}\pi r^2 h$.]

(b) The gradient of a curve which passes through the point (0, 3) is given by

$$\frac{dy}{dx} = 3x^2 - 8x + 5.$$

(i) Determine the equation of the curve. **[3 marks]**

(ii) Find the coordinates of the TWO stationary points of the curve in (b) (i) above and distinguish the nature of EACH point. **[7 marks]**

Total 25 marks

END OF TEST

IF YOU FINISH BEFORE TIME IS CALLED, CHECK YOUR WORK ON THIS TEST.

02134020/CAPE/MS/SPEC

CARIBBEAN EXAMINATIONS COUNCIL

ADVANCED PROFICIENCY EXAMINATION

PURE MATHEMATICS

UNIT 1

ALGEBRA, GEOMETRY AND CALCULUS

SPECIMEN PAPER

PAPER 02

SOLUTIONS AND MARK SCHEMES

SECTION A

(MODULE 1)

Question 1

(a)

p	q	~p (1 mark)	p → q (1 mark)	~p ∨ q (1 mark)
T	T	F	T	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

T = true   F = false
[1 may be used for T and 0 for F]

[3 marks]

(ii) **p → q** and ~p ∨ q are logically equivalent since columns 4 and 5 are identical.

[2 marks]

(iii) $\mathbf{p} \wedge (\mathbf{p} \rightarrow \mathbf{q}) = \mathbf{p} \wedge (\sim\mathbf{p} \vee \mathbf{q})$ (1 mark)

$= (\mathbf{p} \wedge \sim\mathbf{p}) \vee (\mathbf{p} \wedge \mathbf{p})$… distribute ∧ over ∨

$= f \vee (\mathbf{p} \wedge \sim \mathbf{q})$ (1 mark)

$= (\mathbf{p} \wedge \mathbf{q})$ (1 mark)

[3 marks]

(b) (i) $x * y = x + y - 1,\ \forall\ x, y$ in $\mathbb{R}$
$x, y \in \mathbb{R} \Rightarrow x + y \in \mathbb{R}$ (sum of 2 real numbers) (1 mark)
$\Rightarrow x + y - 1 \in \mathbb{R}$ (difference of 2 real numbers)
$\Rightarrow x * y \in \mathbb{R}$ (1 mark)
$\Rightarrow *$ is closed in real numbers (1 mark)

[3 marks]

(ii) $x * y = x + y - 1 = y + x - 1$ (addition is commutative) (1 mark)
$\Rightarrow y * x$ (1 mark)

$\Rightarrow *$ is commutative in $\mathbb{R}$

[2 marks]

(b) (iii) $(x * y) * z = (x + y - 1) * z$ for $x, y, z \in \mathbb{R}$

$= (x + y - 1) + z - 1$
$= x + y + z - 1 - 1$
$= x + y + z - 2$ [2 marks]

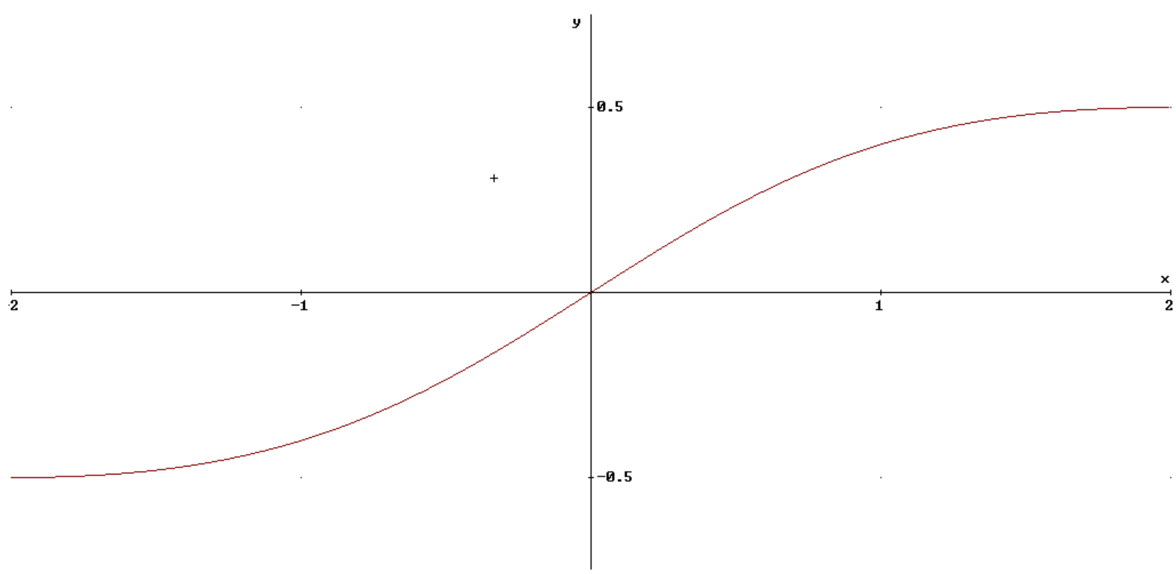
$x * (y * z) = x * (y + z - 1)$
$= x + (y + z - 1) - 1$
$x + y + z - 2$ [1 mark]

$\Rightarrow (x * y) * z = x * (y * z)$ for all $x, y, z \in \mathbb{R}$
$\Rightarrow *$ is associative in $\mathbb{R}$ [1 mark]

(c) (i) $y = \frac{2x}{x^2+4} \Rightarrow y(x^2 + 4) = 2x$ (1 mark)
$\Rightarrow yx^2 - 2x + 4y = 0$

For x real, $(-2)^2 - 4y(4y) \geq 0$ (1 mark)
$\Rightarrow 4y^2 - 1 \leq 0$ (1 mark)
$\Rightarrow (2y - 1)(2y + 1) \leq 0$ (1 mark)
$\Rightarrow -\frac{1}{2} \leq y \leq \frac{1}{2}$ (1 mark)
[5 marks]

(ii) $|x| < 2 \Rightarrow -2 \leq x \leq 2$



[3 marks]

Total 25 marks

<u>Question 2</u>

(a) Let $\mathrm{f}(x) = 2x^3 + px^2 + qx + 2$

(i) $\mathrm{f}(-1) = 0 \Rightarrow -2 + p - q + 2 = 0 \Rightarrow p = q$ (1 mark)

$\mathrm{f}\left(\frac{1}{2}\right) = 0 \Rightarrow \frac{1}{4} + \frac{p}{4} + \frac{q}{2} + 2 = 0 \Rightarrow p + 2q = -9$ (2 mark)

$\Rightarrow p = q = -3$ (1 mark)

(ii) $\mathrm{f}(x) = (2x-1)(x+1)(x-k)$ (1 mark)

$\equiv 2x^3 + px^2\ qx + 2$ (1 mark)

$\Rightarrow k = 2$

$\Rightarrow$ the remaining root is 2 (1 mark)

Alternatively

(a) Let d be the third root of $\mathrm{f}(x) = 0$

(i) Then $-1 \times \frac{1}{2} \times d = -\frac{2}{2} \Rightarrow \frac{d}{2} = 1 \Rightarrow d = 2$ [3 marks]

$d = 2 \Rightarrow -1 + \frac{1}{2} + 2 = -\frac{p}{2} \Rightarrow p = -3$

And $-\frac{1}{2} - 2 + 1 = \frac{q}{2} \Rightarrow q = -3$ [4 marks]

(b) Let P_n be the statement $\sum_{r=1}^{n}(6r + 5) = n\,(3n + 8)$

For $n = 1$, L.H.S. of P_1, is $6 + 5 = 11$ and R.H.S. of P_1

$= 1(3 + 8)$

$= 11$ (1 mark)

So P_n is true for $n = 1$ (1 mark)

Assume that P_n is true for $n = k$, i.e

$11 + 17 + \ldots\ldots + (6k + 5) = k(3k + 8)$ (1 mark)

Then, we need to prove P_n is true for $n = k + 1$

Now $\sum_{r=1}^{k+1}(6r+5) = 11 + 17 + \ldots + (6k + 5) + [6(k + 1) + 5]$ (1 mark)

$= k\,(3k + 8) + [6(k + 1) + 5]$ (1 mark)

$= 3k^2 + 8k + 6k + 6 + 5$

$= 3k^2 + 14k + 11$ (1 mark)

$= (3k + 11)\,(k + 1)$ (1 mark)

$= (k + 1)\,[3\,(k + 1) + 8]$ (1 mark)

Thus, if P_n is true when $n = k$, it is also true with $n = (k + 1)$; (1 mark)

i.e. $\sum_{r=1}^{n}(6r + 5) = n\,(3n + 8)\ \forall\, n \in \mathbb{N}$. (1 mark)

[10 marks]

(c) $e^{2x} + 2e^{-2x} = 3 \Rightarrow e^{2x} + \frac{2}{e^{2x}} = 3$ (1 mark)

$\Rightarrow (e^{2x})^2 - 3e^{2x} + 2 = 0$ (1 mark)

$\Rightarrow (e^{2x} - 2)(e^{2x} - 1) = 0$ (2 mark)

$\Rightarrow e^{2x} = 2$ or $e^{2x} = 1$ (2 mark)

$\Rightarrow 2x = \ln 2$ or $2x = 0$

$\Rightarrow x = \frac{1}{2} \ln 2$ or $x = 0$ (2 mark)

[8 marks]

Alternatively Let $y = e^{2x}$, giving $y^2 - 3y + 2 = 0$ etc.

Total 25 marks

SECTION B
(MODULE 2)

Question 3

(a) (i) LHS $\equiv \dfrac{\sin 3\theta + \sin\theta}{\cos 3\theta + \cos\theta}$ (1 mark)

$$\equiv \frac{2\sin\left(\frac{3\theta+\theta}{2}\right)\cos\left(\frac{(3\theta-\theta)}{2}\right)}{2\cos\left(\frac{3\theta+\theta)}{2}\right)\cos\left(\frac{3\theta-\theta}{2}\right)}$$ (1 mark)

$$\equiv \frac{\sin 2\theta}{\cos 2\theta}$$ (1 mark)

$\equiv \tan 2\theta$ (1 mark)

$\equiv$ RHS

[4 marks]

(ii) $2\sin\left(\frac{3\theta+\theta}{2}\right)\cos\left(\frac{3\theta-\theta}{2}\right) + \sin 2\theta = 0$ (1 mark)

$\Rightarrow 2\sin 2\theta\ \cos\theta + \sin 2\theta\ = 0$ (1 mark)

$\Rightarrow \sin 2\theta\,(2\cos\theta + 1) = 0$ (1 mark)

$\Rightarrow \sin 2\theta = 0$, that is $\theta = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$, or (3 marks)

$\cos\theta = \frac{-1}{2}$, that is $\theta = \frac{2\pi}{3}, \frac{4\pi}{3}$ (1 mark)

[7 marks]

(b) (i) $r = \sqrt{(6^2 + 8^2)} = 10,\ x = \tan^{-1}\left(\frac{3}{4}\right) = 36.9^{\circ}$ (2 marks)

$\therefore\ f(\theta) = 10\cos(\theta - 36.9^{\circ})$ (1 mark)

[3 marks]

(ii) $g(\theta) = \dfrac{10}{10 + 10\cos(\theta - 36.9^{o})}$ (1 mark)

Minimum value of g (θ) occurs when denominator has maximum value i.e.
$\cos(\theta - 36.9) = 1.$ (1 mark)

Min $g(\theta) = \dfrac{10}{10+10} = \dfrac{1}{2}$ occurs
(when $\theta - 36.9 = 0$), when $\theta = 36.9$. (2 marks)

[4 marks]

(c) (i) $\overrightarrow{\mathbf{BC}} = 2\mathbf{j} - 2\mathbf{k}$ and $\overrightarrow{\mathbf{BA}} = 2\mathbf{i} - 2\mathbf{k}$ [2 marks]

(ii) $\mathbf{n}.\overrightarrow{\mathbf{BC}} = (\mathbf{i} + \mathbf{j} + \mathbf{k}).(2\mathbf{j} - 2\mathbf{k}) = 0 + 2 - 2 = 0$ (1 mark)

$\mathbf{n}.\overrightarrow{\mathbf{BA}} = (\mathbf{i} + \mathbf{j} + \mathbf{k}).(2\mathbf{i} - 2\mathbf{k}) = 2 + 0 - 2 = 0$ (1 mark)

So **n** is perpendicular to the plane through A, B and C.

[2 marks]

(iii) Let $\mathbf{r} = x\mathbf{i} + y\mathbf{i} + z\mathbf{k}$ with **r.n** = d (1 mark)

represent the plane through A, B and C.

At the point A, $\mathbf{r} = 2\mathbf{i}$ so $\mathbf{r}.\mathbf{n} = d$ (1 mark)

$\Rightarrow (2\mathbf{i}).(\mathbf{i} + \mathbf{j} + \mathbf{k}) = d \Rightarrow d = 2$

Hence the Cartesian equation of the plane is $(x\mathbf{i} + y\mathbf{i} + z\mathbf{k}).(\mathbf{i} + \mathbf{j} + \mathbf{k}) = 2$ (1 mark)

$\Rightarrow x + y + z = 2$

[3 marks]

[Total 25 marks]

Question 4

(a) L: $x + 2y = 7$ is a tangent to $x^2 + y^2 - 4x = 0$ if $x + 2y = 7$ if L touches the circle

at 2 coincident points (1 mark)

Now, $x = 7 - 2y$ (1 mark)

$\Rightarrow (7 - 2y)^2 - 4(7 - 2y) + y^2 - 1 = 0$ (1 mark)

$\Rightarrow y^2 - 4y + 4 = 0$ (1 mark)

$\Rightarrow (y - 2)^2 = 0$ (1 mark)

$\Rightarrow y = 2(\text{twice})$ (1 mark)

$\Rightarrow$ when $y = 2,\ x = 3$ (1 mark)

So L touches the circle at (3,2) (1 mark)

[8 marks]

(b) (i) Let $Q \equiv$ point diametrically opposite to (3, 2).

The centre of C is (2,0)

so $\dfrac{3+x}{2} = 2,\ x = 1$ (1 mark)

and, $\dfrac{2+y}{2} = 0,\ y = -2$ (1 mark)

$\therefore Q = (1, -2)$

Tangent M at Q: $y + 2 = \dfrac{-1}{2}(x - 1)$ (2 marks)

$2y + x + 3 = 0$ (1 mark)

[5 marks]

(ii) The equation of the diameter is $x + 2y = 2 + 0 = 2$ (1 mark)

This meets C where

$(2 - 2y)^2 + y^2 - 4(2 - 2y) - 1 = 0$ (1 mark)

$\Rightarrow 4 - y + 4y^2 + y^2 - 8 + y - 1 = 0$ (1 mark)

$\Rightarrow 5y^2 - 5 = 0$ (1 mark)

$\Rightarrow y = \pm 1$ [4 marks]

(iii) Coordinates of points of intersection are (1, 0), (–1, 4) [2 marks]

(c) $x(1+t) = t \qquad y(1+t) = t^2$

$\Rightarrow \quad \dfrac{y(1+t)}{x(1+t)} = \dfrac{t^2}{t}$ (1 mark)

$\Rightarrow \quad \dfrac{y}{x} = t$ (1 mark)

$\therefore \quad x = \dfrac{y/x}{1 + y/x}$ (1 mark)

$\Rightarrow \quad x = \dfrac{y}{x+y}$ (1 mark)

$\Rightarrow \quad y = \dfrac{x^2}{1-x}$ (2 marks)

[6 marks]

Total 25 marks

SECTION C
(MODULE 3)

Question 5

(a)

$$\lim_{h \to 0} \frac{h}{\sqrt{x+h} - \sqrt{x}} = \lim_{h \to} \frac{h(\sqrt{x+h} + \sqrt{x})}{(\sqrt{x+h} - \sqrt{x})(\sqrt{x+h} + \sqrt{x})}$$ (1 mark)

$$= \lim_{h \to 0} \frac{h(\sqrt{x+h} + \sqrt{x})}{(x+h) - x}$$ (1 mark)

$$= \frac{\lim}{h \to 0} \frac{h(\sqrt{x+h} + \sqrt{x})}{h}$$ (1 mark)

$$= \lim_{h \to 0} (\sqrt{x+h} + \sqrt{x})$$ (1 mark)

$$= \sqrt{x} + \sqrt{x}$$ (1 mark)

$$= 2\sqrt{x}$$ (1 mark)

[6 marks]

(b) $f''(x) = 18x + 4$

$\Rightarrow$ $f'(x) = 9x^2 + 4x + c$ (1 mark)

$\Rightarrow$ $f(x) = 3x^3 + 2x^2 + cx + d$ (2 marks)

Now $f(2) = 14 \Rightarrow 32 + 2c + d = 14$

$\Rightarrow 2c + d = -18$…. (i) (1 mark)

$f(3) = 74 \Rightarrow 99 + 3c + d = 74$

$\Rightarrow 3c + d = -25$ …..(ii) (1 mark)

From (i) + (ii), $c = -7$ (1 mark)

$d = -4$ (1 mark)

So $f(4) = 3(4^3) + 2(4^2) - 7(4) - 4$

$= 192 + 32 - 32$

$= 192$ (1 mark)

[8 marks]

(c) (i) $y = \dfrac{x}{1+x^2}$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{(1+x^2)1 - x(2x)}{(1+x^2)^2}$$ (2 marks)

$$= \frac{1-x^2}{(1+x^2)^2}$$ (1 mark)

$$= \frac{1}{(1+x^2)^2} - \frac{x^2}{(1+x^2)^2}$$

$$= \frac{1}{(1+x^2)^2} - \left(\frac{x}{1+x^2}\right)^2$$ (1 mark)

$$= \frac{1}{(1+x^2)^2} - y^2$$ (1 mark)

[5 marks]

(ii) $$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = \frac{(1+x^2)^2(-2x) - (1-x^2)2(1+x^2)(2x)}{(1+x^2)^4}$$ (3 marks)

$$= \frac{(1+x^2)(2x)\,[-(1+x^2) - 2 + 2x^2]}{(1+x^2)^4}$$ (2 marks)

$$= \frac{2y(x^2-3)}{(1+x^2)^2}$$ (1 mark)

[6 marks]

Total 25 marks

Question 6

(a) (i) Finding the equation of the tangent PQ.

a) $\dfrac{dy}{dx} = 3x^2$ (1 mark)

$\left(\dfrac{dy}{dx}\right)_{x=3} = 3(3)^2 = 27$ (1 mark)

Equation of tangent:

$y - 27 = 27(x-3)$ (1 mark)

$y = 27x - 54$ (1 mark)

b) Q has coordinates (2, 0) (1 mark)

[5 marks]

(ii) a) Area = $\int_0^3 y\,dx - \dfrac{1}{2}(3-2)(27)$ (2 marks)

$= \dfrac{1}{4}x^4 \Big|_0^3 - \dfrac{27}{2}$ (2 marks)

$= \dfrac{81}{4} - \dfrac{27}{2}$

$= \dfrac{27}{4}$ units^2 (1 mark)

[5 marks]

(b) Required Volume

$= \int_0^3 \pi y^2 dx$ – Volume of the cone with radius 27 units and height 1 unit. (1 mark)

$= \pi \int_0^3 x^6 dx - \dfrac{1}{3}\pi(27)^2$ (1 mark)

$= \pi \dfrac{x^7}{7} \Big|_0^3 - \dfrac{1}{3}\pi(3^6)$ (1 mark)

$= \pi\left(\dfrac{3^7}{7}\right) - \dfrac{1}{3}\pi(3^6)$

$= \dfrac{\pi}{7}\left(3^7 - 7(3^5)\right)$ (1 mark)

$= \dfrac{\pi}{7}\left(2 \times 3^5\right)$ units^3 (1 mark)

[5 marks]

(b) (i) $\dfrac{dy}{dx} = 3x^2 - 8x + 5$

$y = x^3 - 4x^2 + 5x + C$ (1 mark)

substituting; $y = 3$ at $x = 0$

$C = 3$ (1 mark)

$y = x^3 - 4x^2 + 5x + 3$ (1 mark)

[3 marks]

(ii) $\dfrac{dy}{dx} = 0 \Rightarrow 3x^3 - 8x + 5 = 0$ (1 mark)

$(3x - 5)(x - 1) = 0$ (1 mark)

$x = \dfrac{5}{3}; 1$ (1 mark)

$y = \dfrac{131}{27}, 5$ (1 mark)

co-ordinates are $\left(\dfrac{5}{3}, \dfrac{131}{27}\right)$, $(1, 5)$

$\dfrac{d^2y}{dx^2} = 6x - 8$ (1 mark)

$\left(\dfrac{d^2y}{dx^2}\right)_{x=5/3} > 0 \Rightarrow \left(\dfrac{5}{3}, \dfrac{131}{27}\right)_{\max}$ (1 mark)

$\left(\dfrac{d^2y}{dx^2}\right)_{x=1} < 0 \Rightarrow (1, 5)_{\min}$ (1 mark)

[7 marks]

Total 25 marks

FORM TP 02134032/SPEC

C A R I B B E A N   E X A M I N AT I O N S   C O U N C I L

ADVANCED PROFICIENCY EXAMINATION

PURE MATHEMATICS

UNIT 1 – Paper 032

ALGEBRA, GEOMETRY AND CALCULUS

SPECIMEN PAPER

1 hour 30 minutes

The examination paper consists of **THREE** sections: Module 1, Module 2 and Module 3.

Each section consists of 1 question.
The maximum mark for each Module is 20.
The maximum mark for this examination is 60.
This examination consists of 4 printed pages.

READ THE FOLLOWING INSTRUCTIONS CAREFULLY.

1. **DO NOT** open this examination paper until instructed to do so.
2. Answer **ALL** questions from the **THREE** sections.
3. Write your solutions, with full working, in the answer booklet provided.
4. Unless otherwise stated in the question, any numerical answer that is not exact **MUST** be written correct to **THREE** significant figures.

Examination Materials Permitted

Graph paper (provided)
Mathematical formulae and tables (provided) – **Revised 2010**
Mathematical instruments
Silent, non-programmable electronic calculator

DO NOT TURN THIS PAGE UNTIL YOU ARE TOLD TO DO SO

SECTION A (MODULE 1)

Answer this question.

1. (a) **p** and **q** are two given propositions.

(i) State the converse of $\mathbf{p} \rightarrow \mathbf{q}$. **[1 mark]**

(ii) Show that the contrapositive of the inverse of $\mathbf{p} \rightarrow \mathbf{q}$ is the converse of $\mathbf{p} \rightarrow \mathbf{q}$. **[2 marks]**

(b) $f(n) = 2^n + 6^n$

(i) Show that $f(k + 1) = 6f(k) - 4(2^k)$ **[3 marks]**

(ii) Hence, or otherwise, prove by mathematical induction that, for $n \in \mathbb{N}$, $f(n)$ is divisible by 8. **[4 marks]**

(c) (i) On the same diagram, sketch the graphs of $y = x + 2$ and $y = \left|\frac{1}{x-2}\right|$, showing clearly on your sketch the coordinates of any points at which the graphs cross the axes. **[4 marks]**

(ii) Find the range of values of x for which $x + 2 < \left|\frac{1}{x-2}\right|$. **[6 marks]**

Total 20 marks

SECTION B (MODULE 2)

Answer this question.

2. (a) (i) Using $\sin^2\theta + \cos^2\theta \equiv 1$ show that $\text{cosec}^2\theta - \cot^2\theta \equiv 1$. **[2 marks]**

(ii) Hence, or otherwise, prove that $\text{cosec}^4\theta - \cot^4\theta \equiv \text{cosec}^2\theta + \cot^2\theta$.

[2 marks]

(b) A curve, C, has parametric equations

$$x = \sin^2\theta,\ y = 2\tan\theta,\ 0 \le \theta < 90^0.$$

Find the Cartesian equation of C. **[4 marks]**

(c) The line, l_1, has equation $\mathbf{r} = \begin{pmatrix} 2 \\ 3 \\ -4 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$, where λ is a scalar parameter.

The line, l_2, has equation $\mathbf{r} = \begin{pmatrix} 0 \\ 9 \\ -3 \end{pmatrix} + \mu \begin{pmatrix} 5 \\ 0 \\ 2 \end{pmatrix}$, where μ is a scalar parameter.

Given that l_1 and l_2 meet at the point C, find

(i) the coordinates of C **[3 marks]**

(ii) the angle between l_1 and l_2, correct to 2 decimal places. **[4 marks]**

(iii) Show that the vector $\mathbf{n} = 4\mathbf{i} + 3\mathbf{j} - 10\mathbf{k}$ is perpendicular to l_1 and l_2. **[2 marks]**

(iv) Hence find the vector equation of the plane, $\mathbf{r}\,.\,\mathbf{n} = d$, through the point $\begin{pmatrix} 2 \\ 3 \\ -4 \end{pmatrix}$.

[3 marks]

Total 20 marks

SECTION C (MODULE 3)

3. (a) $S_n = 1 \times 1! + 2 \times 2! + 3 \times 3! + \ldots + n \times n! = (n + 1)! - 1$.

Show that $\lim_{n \to \infty} \frac{S_n}{S_{n+1}} = 0$. **[4 marks]**

(b) Using $\lim_{x \to 0} \frac{\sin x}{x} = 1$, differentiate from first principles f $(x) = \cos x$. **[5 marks]**

(c) A circular patch of oil has radius, *r,* metres at time, *t,* hours after it was spilled. At time 2:00 p. m., one hour after the spillage, the radius of the patch of oil is 5 metres. In a model, the rate of increase of r is taken to be proportional to $\frac{1}{r}$.

(i) Form a differential equation for r in terms of t, involving a constant of proportionality, k. **[1 mark]**

(ii) Solve the differential equation in (c) (i) above and hence show that the radius of the patch of oil is proportional to the square root of the time elapsed since the spillage. **[7 marks]**

(iii) Determine the time, to the nearest minute, at which the model predicts that the radius of the patch of oil will be 12 metres. **[3 marks]**

Total 20 marks

END OF TEST

IF YOU FINISH BEFORE TIME IS CALLED, CHECK YOUR WORK ON THIS TEST.

02134032/CAPE/MS/SPEC

CARIBBEAN EXAMINATIONS COUNCIL

ADVANCED PROFICIENCY EXAMINATION

PURE MATHEMATICS

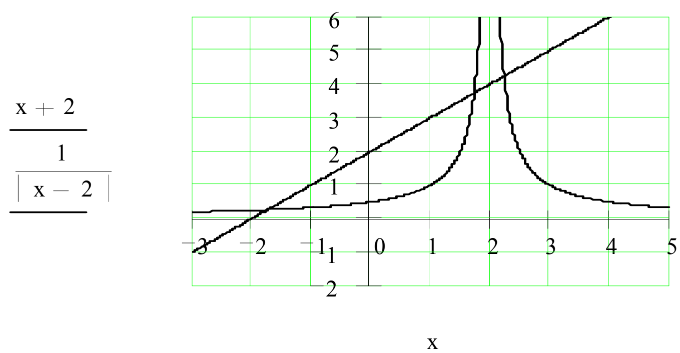
UNIT 1

ALGEBRA, GEOMETRY AND CALCULUS

SPECIMEN PAPER

PAPER 032

SOLUTIONS AND MARK SCHEMES

Question	Details	Marks
1 (a) (i)	The converse of **p** → **q** is **q** → **p**	1
(ii)	The inverse of **p** → **q** is ~ **p** → ~ **q**	1
	The contrapositive of ~ **p** → ~ **q** is ~ ~ **q** → ~ ~ **p** = **q** → **p**	1
(b) (i)	f $(k+1) = 2^{k+1} + 6^{k+1}$ $= 2(2^k) + 6(6^k)$ $= 6(2^k + 6^k) - 4(2^k)$ $= 6\mathrm{f}(k) - 4(2^k)$	1 1 1
(ii)	Assume f (k) is divisible by 8 f $(1) = 2 + 6 = 8$ (true) f $(k+1) = 6$f $(k) - 2 \times 2(2^k) = 6$f $(k) - 2(2^{k+1})$ $= 6$f $(k) - 8\left(\frac{1}{4}2^{k+1}\right)$ which is divisible by 8 Since f(k) is true for $k = 1$ and true for f $(k+1)$ then true for all $n \in \mathbb{N}$	1 1 1 1
(c) (i)	 line , positive gradient, intercepts (-2, 0), (0, 2) curve…..branch > 2 curve…...branch < 2 intercept (0, ½)	1 1 1 1
(ii)	$x + 2 = \dfrac{1}{x-2} \quad x^2 = 5 \quad x = \sqrt{5} \quad (-\sqrt{5}$ not applicable) $x + 2 = \dfrac{1}{2-x} \quad x^2 = 3 \quad x = \sqrt{3} \quad (-\sqrt{3}$ not applicable) $\sqrt{3} < x < \sqrt{5}$ (1) (1)	2 2 1 1
		[20]

S. O. (A) 3, (B) 5, (C) 5, (F) 3

Question	Details	Marks
2 (a) (i)	Dividing throughout by $\sin^2\theta$ gives $\frac{\sin^2\theta}{\sin^2\theta} + \frac{\cos^2\theta}{\sin^2\theta} = \frac{1}{\sin^2\theta}$	1
	$1 + \cot^2\theta = \operatorname{cosec}^2\theta \quad \operatorname{cosec}^2\theta - \cot^2\theta = 1$	1
(ii)	$\operatorname{cosec}^4\theta - \cot^4\theta = (\operatorname{cosec}^2\theta - \cot^2\theta)(\operatorname{cosec}^2\theta + \cot^2\theta)$	1
	$= \operatorname{cosec}^2\theta + \cot^2\theta$	1
(b)	$\frac{1}{x} = \operatorname{cosec}^2\theta \qquad \frac{2}{y} = \cot\theta$	1
	(1) (1)	1
	From (a) (i) $\frac{1}{x} - \frac{4}{y^2} = 1$	1
	$y^2 - 4x = xy^2$	1
(c) (i)	Comparing components of **j**: $3 + 2\lambda = 9 \quad \lambda = 3$	1
	Comparing components of **i**: $2 + 3 = 5\mu \quad \mu = 1$	1
	C (5, 9, −1)	1
(ii)	$\cos\theta = \frac{(\mathbf{i} + 2\mathbf{j} + \mathbf{k}).(5\mathbf{i} + 2\mathbf{k})}{\sqrt{6} \times \sqrt{29}}$ (1 mark each for numerator and denominator)	1
		1
	$\theta = \cos^{-1}\left(\frac{7}{\sqrt{174}}\right)$	1
	$= 57.95^0$	1
(iii)	$(4\mathbf{i} + 3\mathbf{j} - 10\mathbf{k}) . (5\mathbf{i} + 2\mathbf{k}) = 0$	1
	$(4\mathbf{i} + 3\mathbf{j} - 10\mathbf{k}) . (\mathbf{i} + 2\mathbf{j} + \mathbf{k}) = 0$	1
(iv)	**r**.(4**i** + 3**j** – 10**k**) = (2**i** + 3**j** – 4**k**) . (4**i** + 3**j** – 10**k**)	1
	(1) (1)	1
	r.(4i + 3j – 10k) = 57	1
		20

S. O. (A) 5 (ii), 6 (B) 5, (C) 7, 9, 10		
Question	**Details**	**Marks**
3 (a)	$\lim_{n\to\infty}\frac{S_n}{S_{n+1}} = \lim_{n\to\infty}\frac{(n+1)!-1}{(n+2)!-1}$	1
	$= \frac{\lim_{n\to\infty}(n+1)!-1}{\lim_{n\to\infty}(n+2)!-1} = \frac{\lim_{n\to\infty}(n+1)!-1}{\lim_{n\to\infty}(n+2)(n+1)!-1}$ (1 mark each for limit)	1 1
	$= \lim_{n\to\infty}\frac{1}{n+2}$ $= 0$	1
(b)	$\frac{\mathrm{d}}{\mathrm{d}x}\cos x = \lim_{\delta x\to 0}\frac{\cos(x+\delta x)-\cos x}{\delta x}$	1
	$= \lim_{\delta x\to 0}\frac{-2\sin\left(\frac{x+\delta x+x}{2}\right)\sin\left(\frac{x+\delta x-x}{2}\right)}{\delta x}$	1
	$= \lim_{\delta x\to 0}\frac{-2\sin\left(x+\frac{\delta x}{2}\right)\sin\left(\frac{\delta x}{2}\right)}{\delta x}$	1
	Given $\lim_{\delta x\to 0}\frac{\sin x}{x} = 1$ $\quad \frac{1}{2}\lim_{\delta x\to 0}\frac{\sin\frac{\delta x}{2}}{\left(\frac{\delta x}{2}\right)} = \frac{1}{2}$	1
	$\lim_{\delta x\to 0}\frac{-2\sin\left(x+\frac{\delta x}{2}\right)}{2} = -\sin x$	1
(c) (i)	$\frac{\mathrm{d}r}{\mathrm{d}t} = \frac{k}{r}$	1
(ii)	$\int r\,\mathrm{d}r = \int k\,\mathrm{d}t$	1
	$\frac{1}{2}r^2 = kt + A$ (1) (1)	1 1
	$r = 0$ $t = 0$ gives $A = 0$ $r = 5$ $t = 1$ gives $k = \frac{25}{2}$ $r^2 = 25t$ $\quad r = 5\sqrt{t}$	1 1 1 1

Question	Details	Marks
3 (c) (iii)	$12 = 5\sqrt{t}$ (1) $\quad t = \left(\frac{12}{5}\right)^2 = 5.76\text{h} = 5$ hrs 46 mins (1) Time when radius is 12 metres is 6:46 p. m	1 1 1 20
	S. O. (A) 3, 4, 5, (B) 2 (iv), (C) 9 (i) (ii)	

TEST CODE 02234010/SPEC

FORM TP 02134010/SPEC

C A R I B B E A N   E X A M I N A T I O N S   C O U N C I L

ADVANCED PROFICIENCY EXAMINATION

SPECIMEN PAPER
MULTIPLE CHOICE QUESTIONS
FOR

PURE MATHEMATICS

UNIT 2 – Paper 01

COMPLEX NUMBERS, ANALYSIS AND MATRICES

90 minutes

READ THE FOLLOWING INSTRUCTIONS CAREFULLY.

1. This test consists of 45 items. You will have 90 minutes to answer them.

2. Each item in this test has four suggested answers lettered (A), (B), (C), (D). Read each item you are about to answer and decide which choice is best.

3. Look at the sample item below.

   Sample Item
   If the function $f(x)$ is defined by $f(x)$ cos x then $f(x)$ is

   (A) $-\frac{1}{2\sqrt{x}} \sin\sqrt{x}$

   (B) $-\frac{1}{2} \sin\sqrt{x}$

   (C) $\frac{1}{\sqrt{x}} \sin\sqrt{x}$

   (D) $\frac{1}{2\sqrt{x}} \sin\sqrt{x}$

   Sample Answer

   A B C

   The best answer to this item is "$\frac{1}{2\sqrt{x}} \sin\sqrt{x}$", so answer space (D) has been shaded.

4. You may do any rough work in this booklet.

5. The use of silent, non-programmable scientific calculators is allowed.

Examination Materials Permitted

A list of mathematical formulae and tables (provided) – **Revised 2010**

DO NOT TURN THIS PAGE UNTIL YOU ARE TOLD TO DO SO

1. If z and z^* are two conjugate complex numbers, where $z = x + \mathrm{i}y$, $x, y \in \mathbb{R}$, then $z\,z^* =$

(A) $x^2 + y^2$
(B) $x^2 - y^2 - 2xy\mathrm{i}$
(C) $x^2 - y^2$
(D) $x^2 + y^2 - 2xy\mathrm{i}$

2. If $|\, z + \mathbf{i} \,| = |\, z + 1|$, where z is a complex number, then the locus of z is

(A) $y = 0$
(B) $y = 1$
(C) $y = x$
(D) $y = \dfrac{x}{2}$

3. Given that $z + 3z^* = 12 + 8\mathrm{i}$, then $z =$

(A) $-3 - 4\mathrm{i}$
(B) $3 - 4\mathrm{i}$
(C) $3 + 4\mathrm{i}$
(D) $-3 + 4\mathrm{i}$

4. One root of a quadratic equation with real coefficients is $2 - 3\mathrm{i}$. The quadratic equation is

(A) $x^2 + 4x + 13 = 0$
(B) $x^2 - 4x - 13 = 0$
(C) $x^2 + 4x - 13 = 0$
(D) $x^2 - 4x + 13 = 0$

5. If $z = \cos\theta + \mathrm{i}\sin\theta$, then $z^4 + \dfrac{1}{z^4} =$

(A) $2\cos 4\theta$
(B) $2\mathrm{i}\sin 4\theta$
(C) $\cos 4\theta + \mathrm{i}\sin 4\theta$
(D) $4\cos\theta - \mathrm{i}\,(4\sin\theta)$

6. The gradient of the normal to the curve with the equation $xy^3 + y^2 + 1 = 0$ at the point $(2, -1)$ is

(A) -4

(B) $-\dfrac{1}{4}$

(C $\dfrac{1}{4}$

(D) 4

7. If $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{5x}{y}$, then $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} =$

(A) $\dfrac{5}{y^3} - \dfrac{25x^2}{y^2}$

(B) $\dfrac{5}{y} - \dfrac{25x^2}{y^3}$

(C) $\dfrac{1}{y} - \dfrac{25}{xy}$

(D) $\dfrac{5}{xy} - \dfrac{25}{x^3y^3}$

8. The curve C is given by the parametric equations $x = t + \mathrm{e}^{-t}$, $y = 1 - \mathrm{e}^{-t}$. The gradient function for C at the point (x, y) is given as

(A) $\dfrac{1}{1 - \mathrm{e}^t}$

(B) $\dfrac{1}{\mathrm{e}^t - 1}$

(C) $\dfrac{-1}{1 + \mathrm{e}^t}$

(D) $\dfrac{1}{\mathrm{e}^t + 1}$

GO ON TO THE NEXT PAGE

9. Given $y = a\ arcos(ax)$, where a is a constant, $\frac{dy}{dx} =$

(A) $\frac{a^2}{\sqrt{(1 - a^2x^2)}}$

(B) $-\frac{1}{\sqrt{(1 - a^2x^2)}}$

(C) $-\frac{a^2}{\sqrt{(1 - a^2x^2)}}$

(D) $\frac{1}{\sqrt{(1 - a^2x^2)}}$

10. Given that $f(x, y, z) = x^2y + y^2z - z^2x$ then $\frac{\partial f}{\partial y} =$

(A) $x^2y + 2yz$
(B) $x^2 + 2yz$
(C) $x^2 + y^2$
(D) $x^2 + y^2 + z^2$

11. $\int \frac{x^3}{(x^2 - 3x + 2)}\,dx$ may be expressed as

(A) $\int \left(\frac{Px + Q}{x^2 - 3x + 2}\right) dx$

(B) $\int \left(\frac{P}{x - 1} + \frac{Q}{x - 2}\right) dx$

(C) $\int \left(x + 3 + \frac{Px}{x^2 - 3x + 2}\right) dx$

(D) $\int \left(x + 3 + \frac{P}{x - 1} + \frac{Q}{x - 2}\right) dx$

12. $\int \frac{1}{\sqrt{(a^2 - x^2)}}\,dx =$

(A) $a\sqrt{(a^2 - x^2)} + C$

(B) $\arcsin(x^2) + C$

(C) $\arcsin\left(\frac{x}{a}\right) + C$

(D) $a \arcsin(ax) + C$

13. Given that $y = \frac{\pi}{2} - x$, then $\int_0^{\frac{\pi}{2}} \sin^2 x\,dx =$

(A) $\int_0^{\frac{\pi}{2}} \cos^2 y\,dy$

(B) $\int_0^{\frac{\pi}{2}} \sin^2 y\,dy$

(C) $\int_0^{\frac{\pi}{2}} (\sin x + \cos x)\,dx$

(D) $\int_0^{\frac{\pi}{2}} (\sin y + \cos y)\,dy$

14. Given $I_n = \int \tan^n x\,dx, for\ n > 2, I_n =$

(A) $\frac{1}{n-1} \tan^{n-1} x + I_{n-2}$

(B) $\frac{1}{n-1} \tan^{n-1} x \sec^2 x - I_{n-2}$

(C) $\tan^{n-1} x - I_{n-2}$

(D) $\frac{1}{n-1} \tan^{n-1} x - I_{n-2}$

15. The value of $\int_0^{\pi/2} x \cos \mathrm{d}x$ is

(A) 1

(B) $\frac{\pi}{2}$

(C) $\frac{\pi}{2} - 1$

(D) $\frac{\pi}{2} + 1$

16. Given that a sequence of positive integers $\{U_n\}$ is defined by $U_1 = 2$ and $U_{n+1} = 3U_n + 2$, then $U_n =$

(A) $3n - 1$
(B) $3^n + 1$
(C) $3^n - 1$
(D) $3n + 2$

17. The sequence $a_n = \frac{3n^2 - n + 4}{2n^2 + 1}$

(A) converges
(B) diverges
(C) is periodic
(D) is alternating

18. The nth term of a sequence is given by $u_n = 9 - 4\left(\frac{1}{2}\right)^{n-1}$. The 5th term of the sequence is

(A) $\frac{9}{4}$

(B) $\frac{35}{4}$

(C) $\frac{37}{4}$

(D) $\frac{71}{8}$

19. $\sum_{r=1}^{m-1} 3\left(\frac{1}{2}\right)^r =$

(A) $3 - 3 \times 2^{-m}$
(B) $3 - 3 \times 2^{(1-m)}$
(C) $6 - 3 \times 2^{(m-1)}$
(D) $6 - 3 \times 2^{(1-m)}$

20. Given that $\sum_{r=1}^{n} u_n = 5n + 2n^2$, then $u_n =$

(A) $4n + 3$
(B) $5n + 2$
(C) $2n^2 + n - 3$
(D) $4n^2 + 4n + 7$

21. The sum to infinity, $S(x)$, of the series $1 + \left(\frac{2}{1+x}\right) + \left(\frac{2}{1+x}\right)^2 + \left(\frac{2}{1+x}\right)^3 +$ is

(A) 1

(B) $\frac{x+2}{x+1}$

(C) $\frac{x+1}{x-1}$

(D) $1 + \left(\frac{2}{x+1}\right)^n$

22. The Maclaurin's series expansion for $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \ldots$ has a general term BEST defined as

(A) $(-1)^n \frac{x^{n+1}}{(n+1)!}$

(B) $(-1)^{n+1} \frac{x^{n+1}}{(n+1)!}$

(C) $(-1)^n \frac{x^{2n+1}}{(n+1)!}$

(D) $(-1)^{n+1} \frac{x^{2n-1}}{(2n-1)!}$

GO ON TO THE NEXT PAGE

23. The first 2 non-zero terms of the expansion of sin $(x + \pi/6)$ are

(A) $\frac{1}{2} + \frac{\sqrt{3}}{\text{x}}x$

(B) $\frac{1}{2} - \frac{\sqrt{3}}{2}x$

(C) $\frac{1}{2} + \frac{1}{2}x$

(D) $\frac{\sqrt{3}}{2} + \frac{1}{2}x$

24. If ${}^{n}C_{r} = {}^{n-1}C_{r-1}$, then

(A) $n = r$
(B) $n - r = 1$
(C) $r - n = 1$
(D) $n - 1 = r + 1$

25. If $\left(2x^2 - \frac{2}{x}\right)^6 = ... + k + ...$, where k is independent of x, then $k =$

(A) –960
(B) –480
(C) 480
(D) 960

26. An investment prospectus offers that for an initial deposit of \$5 000 at January 1st an interest rate of 3% will be applied at December 31st on the opening balance for the year. Assuming that no withdrawals are made for any year, the value of the investment n years after the initial deposit is given by

(A) $(5\,000)\,(1.03)^n$
(B) $(5\,000)\,(0.03)^n$
(C) $(5\,000)\,(1.03)^{n-1}$
(D) $(5\,000)\,(0.03)^{n-1}$

27. The coefficient of x^3 in the expansion of $\left(1 + x + x^2\right)^5$ is

(A) 5
(B) 20
(C) 30
(D) 40

28. The equation $\sin x^2 + 0.5x - 1 = 0$ has a real root in the interval

(A) (0.8, 0.9)
(B) (0.7, 0.8)
(C) (0.85, 0.9)
(D) (0.9, 0.10)

29. $f(x) = x^3 - \frac{7}{x} + 2$, $x > 0$. Given that $f(x)$ has a real root α in the interval (1.4, 1.5), using the interval bisection once α lies in the interval

(A) (1.45, 1.5)
(B) (1.4, 1.45)
(C) (1.425, 1.45)
(D) (1.4, 1.425)

30. $f(x) = x^3 - x^2 - 6$. Given that $f(x) = 0$ has a real root α in the interval [2.2, 2.3], applying linear interpolation once on this interval an approximation to α, correct to 3 decimal places, is

(A) 2.216
(B) 2.217
(C) 2.218
(D) 2.219

31. Taking 1.6 as a first approximation to α, where the equation $4 \cos x + e^{-x} = 0$ has a real root α in the interval (1.6, 1.7), using the Newton-Raphson method a second approximation to α (correct to 3 decimal places) ≈

(A) 1.602
(B) 1.620
(C) 1.622
(D) 1.635

32. $f(x) = 3x^3 - 2x - 6$. Given that $f(x) = 0$ has a real root, α, between $x = 1.4$ and $x = 1.45$, starting with $x_0 = 1.43$ and using the iteration $x_{n+1} = \sqrt{\left(\frac{2}{x_n} + \frac{2}{3}\right)}$, the value of x_1 correct to 4 decimal places is

(A) 1.4369
(B) 1.4370
(C) 1.4371
(D) 1.4372

33. Ten cards, each of a different colour, and consisting of a red card and a blue card, are to be arranged in a line. The number of different arrangements in which the red card is not next to the blue card is

(A) $9! - 2 \times 2!$
(B) $10! - 9! \times 2!$
(C) $10! - 2! \times 2!$
(D) $8! - 2! \times 2!$

34. The number of ways in which all 10 letters of the word **STANISLAUS** can be arranged if the **S**s must all be together is

(A) $\frac{8! \times 3!}{2!}$

(B) $8! \times 3!$

(C) $\frac{8!}{3!}$

(D) $\frac{8!}{2!}$

35. A committee of 4 is to be chosen from 4 teachers and 4 students. The number of different committees that can be chosen if there must be at least 2 teachers is

(A) 36
(B) 45
(C) 53
(D) 192

36. A and B are two events such that $P(A) = p$ and $P(B) = \frac{1}{3}$. The probability that neither occurs is $\frac{1}{2}$. If A and B are mutually exclusive events then $p =$

(A) $\frac{5}{6}$

(B) $\frac{2}{3}$

(C) $\frac{1}{5}$

(D) $\frac{1}{6}$

37. On a randomly chosen day the probability that Bill travels to school by car, by bicycle or on foot is $\frac{1}{2}$, $\frac{1}{6}$ and $\frac{1}{3}$ respectively. The probability of being late when using these methods of travel is $\frac{1}{5}$, $\frac{2}{5}$ and $\frac{1}{10}$ respectively. The probability that on a randomly chosen day Bill travels by foot and is late is

(A) $\frac{1}{30}$

(B) $\frac{1}{10}$

(C) $\frac{3}{10}$

(D) $\frac{13}{30}$

38. Given $\begin{vmatrix} 6 & 0 & 1 \\ 7 & 7 & 0 \\ 0 & -12 & x \end{vmatrix} = 0$, the value of x is

(A) –2
(B) 2
(C) 7
(D) 12

Items **39 – 40** refer to the matrix below.

$$A = \begin{pmatrix} 2 & -7 & 8 \\ 3 & -6 & -5 \\ 4 & 0 & -1 \end{pmatrix}$$

39. The transpose of matrix, A, results in $|A|$ being

(A) 0
(B) squared
(C) negative
(D) unchanged

40. The matrix resulting from adding Row 1 to Row 2 is

(A) $\begin{pmatrix} -1 & -1 & 13 \\ 3 & -6 & -5 \\ 4 & 0 & -1 \end{pmatrix}$

(B) $\begin{pmatrix} 2 & -7 & 8 \\ -1 & -1 & -13 \\ 4 & 0 & -1 \end{pmatrix}$

(C) $\begin{pmatrix} -5 & 7 & 8 \\ -3 & -6 & -5 \\ 4 & 0 & -1 \end{pmatrix}$

(D) $\begin{pmatrix} -5 & 5 & -8 \\ 3 & -3 & 5 \\ 4 & -0 & 1 \end{pmatrix}$

41. Given $A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 6 \end{pmatrix}$ and $B = \begin{pmatrix} -4 & 0 & 2 \\ 0 & 6 & -4 \\ 2 & -4 & 2 \end{pmatrix}$, by considering AB, then $A^{-1} =$

(A) $2B$

(B) B

(C) $\frac{1}{2}B$

(D) $\frac{1}{2}AB$

42. The general solution of the differential equation $\sin x\frac{dy}{dx} - y\cos x = \sin 2x \sin x$ is found by evaluating

(A) $\int \frac{d}{dx} y \sin x \, dx = \int 2\cos x \, dx$

(B) $\int \frac{d}{dx} \frac{y}{\sin x} \, dx = \int 2\cos x \, dx$

(C) $\int \frac{d}{dx} \frac{y}{\sin x} \, dx = \int \sin 2x \, dx$

(D) $\int \frac{d}{dx} \frac{y}{\sin x} \, dx = \int \cos x \, dx$

43. The general solution of the differential equation $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 3e^x$ is of the form

(A) $y = Ae^x + Be^{2x} + ke^x$
(B) $y = Ae^x + Be^{2x} - 3e^x$
(C) $y = Ae^{-x} + Be^{-2x} + kxe^x$
(D) $y = Ae^x + Be^{2x} + kxe^x$

44. A particular integral of the differential equation $\frac{d^2y}{dx^2} + 25y = 3\cos 5x$ is of the form $y = \lambda x \sin 5x$. The general solution of the differential equation is

(A) $y = A\cos 5x - B\sin 5x - \lambda x\sin 5x$
(B) $y = A\cos 5x + B\sin 5x + \lambda x\sin 5x$
(C) $y = A\cos 5x + B\sin 5x - \lambda x\sin 5x$
(D) $y = A\cos 5x - B\sin 5x + \lambda x\sin 5x$

45. The general solution of the differential equation $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = 2x^2 + x - 1$ is

(A) $y = e^{2x}(A + Bx) + ax^2 + bx + c$
(B) $y = e^{-2x}(A + Bx) + ax^2 + bx + c$
(C) $y = e^{2x}(A + Bx) + 2x^2 + x - 1$
(D) $y = e^{2x}(A - Bx) + ax^2 + bx + c$

END OF TEST

IF YOU FINISH BEFORE TIME IS CALLED, CHECK YOUR WORK ON THIS TEST

C A R I B B E A N   E X A M I N A T I O N S   C O U N C I L

CARIBBEAN ADVANCED PROFICIENCY EXAMINATION

PURE MATHEMATICS

UNIT 2, PAPER 01

MARK SCHEME

Key

Unit 2 Paper 01

Module	Item	Key	S.O.	Module	Item	Key	S.O.
1	1	A	A2	3	31	B	D5
	2	C	A11		32	C	D6
	3	B	A5		33	D	A2
	4	D	A3		34	D	A3
	5	A	A12		35	C	A4
	6	A	B4		36	D	A14
	7	B	B9		37	A	A16
	8	B	B3		38	B	B2
	9	C	B8		39	D	B1
	10	B	B6		40	A	B3
	11	D	C3		41	C	B7
	12	C	C9		42	B	C1
	13	A	C7		43	D	C3(ii)(i)
	14	D	C10		44	B	C3(iii)(iii)
	15	C	C8		45	A	C3(i)(ii)
2	16	C	A1				
	17	A	A3				
	18	B	A2				
	19	D	B4				
	20	A	B3				
	21	C	B6				
	22	D	B8				
	23	A	B9				
	24	A	C1				
	25	D	C3				
	26	C	C4				
	27	C	C3				
	28	A	D1				
	29	B	D2				
	30	C	D3				

FORM TP 02234020/SPEC

C A R I B B E A N   E X A M I N A T I O N S   C O U N C I L

ADVANCED PROFICIENCY EXAMINATION

PURE MATHEMATICS

UNIT 2 – Paper 02

COMPLEX NUMBERS, ANALYSIS AND MATRICES

SPECIMEN PAPER

2 hours 30 minutes

The examination paper consists of **THREE** sections: Module 1, Module 2 and Module 3.

Each section consists of 2 questions.
The maximum mark for each Module is 50.
The maximum mark for this examination is 150.
This examination consists of 5 printed pages.

READ THE FOLLOWING INSTRUCTIONS CAREFULLY.

1. **DO NOT** open this examination paper until instructed to do so.
2. Answer **ALL** questions from the **THREE** sections.
3. Write your solutions, with full working, in the answer booklet provided.
4. Unless otherwise stated in the question, any numerical answer that is not exact **MUST** be written correct to **THREE** significant figures.

Examination Materials

Graph paper (provided)
Mathematical formulae and tables (provided) – **Revised 2010**
Mathematical instruments
Silent, non- programmable, electronic calculator

DO NOT TURN THIS PAGE UNTIL YOU ARE TOLD TO DO SO

SECTION A (MODULE 1)

Answer BOTH questions.

1. (a) (i) Express the complex number $\frac{4 - 2\mathbf{i}}{1 - 3\mathbf{i}}$ in the form of a + ib where a and b are real numbers. **[4 marks]**

(ii) Show that the argument of the complex number in (a) (i) above is $\frac{\pi}{4}$. **[1 mark]**

(b) (i) Find the complex number $u = x + \mathbf{i}y$, $x, y \in \mathbf{I}$, such that $u^2 = -5 + 12\mathbf{i}$. **[8 marks]**

(ii) Hence, solve the equation that $z^2 + \mathbf{i}z + (1 - 3\mathbf{i}) = 0$. **[6 marks]**

(c) Find the complex number that $z = a + \mathbf{i}b$ such that

$$(1 + 3\mathbf{i})z + (4 - 2\mathbf{i})\, z^* = 10 + 4\mathbf{i}$$ **[6 marks]**

Total 25 marks

2. (a) Find $\int e^{3x} \sin 2x \, dx$. **[7 marks]**

(b) (i) a) Find $\frac{dy}{dx}$ when $y = \tan^{-1}(3x)$. **[4 marks]**

b) Hence, find $\int \frac{(x + 2)}{1 + 9x^2} \, dx$. **[4 marks]**

(ii) Show that if $y = \frac{ln(5x)}{x^2}$ then $\frac{dy}{dx} = \frac{1 - ln(25x^2)}{x^3}$ **[5 marks]**

(c) Let $f(x, y) = x^2 + y^2 - 2xy$.

(i) Find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ **[2 marks]**

(ii) Show that

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = 2f(x, y)$$ **[3 marks]**

Total 25 marks

SECTION B (MODULE 2)

Answer BOTH questions.

3. (a) (i) Find constants A and B such that

$$\frac{1}{(2r-1)(2r+1)} \equiv \frac{A}{2r-1} + \frac{B}{2r+1}.$$ **[5 marks]**

(ii) Hence, find the value of S where

$$S = \sum_{r=1}^{n} \frac{1}{(2r-1)(2r+1)}.$$ **[5 marks]**

(iii) Deduce the sum to infinity of S. **[3 marks]**

(b) (i) Find the r^{th} term of the series

$$1(2) + 2(5) + 3(8) + \ldots$$ **[2 marks]**

(ii) Prove, by Mathematical Induction, that the sum to n terms of the series in (b) (i) above is $n^2(n + 1)$. **[10 marks]**

Total 25 marks

4. (a) Given the series

$$\frac{1}{2} + \frac{1}{2^4} + \frac{1}{2^7} + \frac{1}{2^{10}} + \ldots.$$

(i) show that the series is geometric **[3 marks]**

(ii) find the sum of the series to n terms. **[4 marks]**

(b) Use Maclaurin's Theorem to find the FIRST three non-zero terms in the power series expansion of $\cos 2x$. **[7 marks]**

(c) (i) Expand $\sqrt{\left(\frac{1+x}{1-x}\right)}$ up to and including the term in x^3 stating the values of x for which the expansion is valid. **[7 marks]**

(ii) By taking $x = 0.02$ find an approximation for $\sqrt{51}$, correct to 5 decimal places. **[4 marks]**

Total 25 marks

SECTION C (MODULE 3)

Answer BOTH questions.

5. (a) Two cards are drawn without replacement from ten cards which are numbered 1 to 10. Find the probability that

(i) the numbers on **BOTH** cards are even **[4 marks]**

(ii) the number on one card is odd and the number on the other card is even. **[4marks]**

(b) A journalist reporting on criminal cases classified 150 criminal cases by the age (in years) of the criminal and by the type of crime committed, violent or non-violent. The information is presented in the table below.

Type of Crime	**Age (in years)**		
	Less than 20	**20 to 39**	**40 or older**
Violent	27	41	14
Non-violent	12	34	22

What is the probability that a case randomly selected by the journalist

(i) is a violent crime? **[2 marks]**

(ii) was committed by someone LESS than 40 years old? **[4 marks]**

(iii) is a violent crime OR was committed by a person LESS than 20 years old? **[3 marks]**

(c) On a particular weekend, 100 customers made purchases at Green Thumb Garden supply store. Of these 100 customers;

30 purchased tools
45 purchased fertilizer
50 purchased seeds
15 purchased seeds and fertilizer
20 purchased seeds and tools
15 purchased tools and fertilizer
10 purchased tools, seeds and fertilizer.

(i) Represent the above information on a Venn diagram. **[4 marks]**

(ii) Determine how many customers purchased:

a) only tools
b) seeds and tools but not fertilizer,
c) tools and fertilizer but not seeds,
d) neither seeds, tools, nor fertilizer. **[4 marks]**

Total 25 marks

6. (a) Solve for x the following equation

$$\begin{vmatrix} 5 & x & 3 \\ x+2 & 2 & 1 \\ -3 & 2 & x \end{vmatrix} = 0$$

[8 marks]

(b) Solve the first order differential equation

$$y \tan x \frac{\mathrm{d}y}{\mathrm{d}x} = (4 + y^2) \sec^2 x$$

[4 marks]

(c) Given that $y = u \cos 3x + v \sin 3x$ is a particular integral of the differential equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 4\frac{\mathrm{d}y}{\mathrm{d}x} + 3y = 30 \sin 3x,$$

find

(i) the values of the constants u and v,

[8 marks]

(ii) the general solution of the differential equation. **[5 marks]**

Total 25 marks

END OF TEST

IF YOU FINISH BEFORE TIME IS CALLED, CHECK YOUR WORK ON THIS TEST.

02234020/CAPE/MS/SPEC

CARIBBEAN EXAMINATIONS COUNCIL

ADVANCED PROFICIENCY EXAMINATION

PURE MATHEMATICS

SPECIMEN PAPER

UNIT 2

COMPLEX NUMBERS, ANALYSIS AND MATRICES

PAPER 02

SOLUTIONS AND MARK SCHEMES

SECTION A

(MODULE 1)

Question 1

(a) (i) $\frac{4-2i}{1-3i} = \frac{(4-2i)(1+3i)}{1-3i)(1+3i)}$

$= \frac{4+12i - 2i - 6i^2}{1+9}$ (1 mark)

$= \frac{4+10i+6}{10}$ (1 mark)

$= \frac{10+10i}{10}$ (1 mark)

$= 1+i$ (1 mark)

[4 marks]

(ii) arg is $\tan^{-1}(1)\ ) = \frac{\pi}{4}$ (1 mark)

[1 mark]

(b) (i) Let $u = x + \mathrm{i}\, y$, where x, y are real nos.

$u^2 = -5 + 12\mathrm{i} \ \Rightarrow (x + \mathrm{i}y)^2 = -5 + 12\mathrm{i}$

$\Rightarrow x^2 - y^2 + 2\mathrm{i}xy = -5 + 12\mathrm{i}$ (1 mark)

$\Rightarrow x^2 - y^2 = -5, \quad 2xy = 12$ (1 mark)

$\Rightarrow x^2 - \left(\frac{6}{x}\right)^2 = -5, \quad y = \frac{6}{x}$

$\Rightarrow x^2 - \frac{36}{x^2} = -5$ (1 mark)

$\Rightarrow (x^2)^2 + 5x^2 - 36 = 0$ (1 mark)

$\Rightarrow (x^2 + 9)(x^2 - 4) = 0$ (1 mark)

$\Rightarrow x^2 = -9(\text{inadmissable}), x^2 = 4$ (1 mark)

$\Rightarrow x = \pm 2, y = \mp 3$ (1 mark)

$\Rightarrow = 2 - 3\mathrm{i} \text{ or} - 2 + 3\mathrm{i}$ (1 mark)

[8 marks]

(b) (ii) $z^2 + \mathrm{i}z + (1 - 3\mathrm{i}) = 0 \Rightarrow z = \frac{-\mathrm{i}\pm\sqrt{\mathrm{i}^2-4(1-3\mathrm{i})}}{2}$ (1 mark)

$\Rightarrow z\frac{-\mathrm{i}\pm\sqrt{-1-4+12\mathrm{i}}}{2}$ (1 mark)

$\Rightarrow z\frac{-\mathrm{i}\pm\sqrt{-5+12\mathrm{i}}}{2}$ (1 mark)

$\Rightarrow z\frac{-\mathrm{i}\pm2-3\mathrm{i}}{2}$ (1 mark)

$\Rightarrow z\frac{2-4\mathrm{i}}{2}$ or $\frac{2+2\mathrm{i}}{2}$ (1 mark)

$\Rightarrow z = 1 - 2\mathrm{i}$ or $- 1 + \mathrm{i}$ (1 mark)

[6 marks]

(c) $(1 + 3\mathrm{i})z + (4 - 2\mathrm{i})\mathrm{z} = 10 + 4\mathrm{i}$, and $z = a + \mathrm{i}b$

$\Rightarrow (1 + 3\mathrm{i})(a + \mathrm{i}b) + (4 - 2\mathrm{i})(a - \mathrm{i}b) = 10 + 4\mathrm{i}$ (1 mark)

$\Rightarrow (a - 3b) + \mathrm{i}(3a + b) + (4a - 2b) + \mathrm{i}(-4b - 2a) = 10 + 4\mathrm{i}$ (1 mark)

$\Rightarrow a - 3b + 4a - 2b = 10$ and $3a + b - 4b - 2a = 4$ (1 mark)

$\Rightarrow 5a - 5b = 10$ and $a - 3b = 4$ (1 mark)

$\Rightarrow a = 1, b = -1$ (1 mark)

$\Rightarrow z = 1 - \mathrm{i}$ (1 mark)

[6 marks]

Total 25 marks

Question 2

(a) Let $\mathrm{I} = \int \mathrm{e}^{3x} \sin 2x \, \mathrm{d}x$

$= \frac{1}{3}\mathrm{e}^{3x} \sin 2x - \int \frac{\mathrm{e}^{3x}}{3} \, (2\cos 2x) \, \mathrm{d}x$ (2 marks)

$= \frac{1}{3}\mathrm{e}^{3x} \sin 2x - \frac{2}{3}\int \mathrm{e}^{3x} \, (2\cos 2x) \, \mathrm{d}x$

$= \frac{1}{3}\mathrm{e}^{3x} \sin 2x - \frac{2}{3}\left[\frac{1}{3}\mathrm{e}^{3x} \cos 2x + \int \frac{\mathrm{e}^{3x}}{3} \, (2\sin 2x) \, \mathrm{d}x\right]$ (2 marks)

$= \frac{1}{3}\mathrm{e}^{3x} \sin 2x - \frac{2}{9} \mathrm{e}^{3x}\cos 2x - \frac{4}{9}\int \mathrm{e}^{3x} \sin 2x \mathrm{d}x$

$= \frac{1}{3}\mathrm{e}^{3x} \sin 2x - \frac{2}{9} \mathrm{e}^{3x}\cos 2x - \frac{4}{9}I$ (1 mark)

$\Rightarrow I + \frac{4}{9}I = \frac{1}{9}(3 \sin 2x - 2\cos 2x)$ (1 mark)

$\Rightarrow I = \frac{1}{13}(3 \sin 2x - 2\cos 2x) + \text{constant}$ (1 mark)

[7 marks]

Alternatively

$\int \mathrm{e}^{3x} \, \mathrm{e}^{2\mathrm{i}x} \mathrm{d}x = \int \mathrm{e}^{(3+2\mathrm{i})x} \, \mathrm{d}x$ (2 marks)

$\Rightarrow \operatorname{Im} \left[\frac{\mathrm{e}^{(3+2\mathrm{i})x}}{3+2\mathrm{i}}\right] + \text{ constant}$ (2 marks)

$\Rightarrow \int \mathrm{e}^{3x} \sin 2x \mathrm{d}x = \operatorname{Im} \frac{(3-2\mathrm{i})}{13} \, \mathrm{e}^{3x}(\cos 2x + \mathrm{i}\sin 2x)$ (2 marks)

$\frac{\mathrm{e}^{3x}}{13}(3 \sin 2x - 2\cos 2x) + \text{const.}$ (1 mark)

[7 marks]

(b) (i) a) $y = \tan^{-1}(3x) \Rightarrow \tan y = 3x$ (1 mark)

$\Rightarrow \sec^2 y \, \frac{\mathrm{d}y}{\mathrm{d}x} = 3$

$\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{3}{\sec^2 y}$ (1 mark)

$\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{3}{1+\tan^2 y}$ (1 mark)

$\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{3}{1 + 9x^2}$ (1 mark)

[4 marks]

b) $\int \frac{x+2}{1 + 9x^2} \mathrm{d}x = \int \frac{x}{1 + 9x^2} \, \mathrm{d}x + 2\int \frac{1}{1 + 9x^2} \, \mathrm{d}x$ (1 mark)

$= \frac{1}{18}\ln\,(1 + 9x^2) + \frac{2}{3} \, \tan^{-1}(3x) + \text{constant}$ (3 marks)

[4 marks]

(b) (ii) $y = \frac{\ln(5x)}{x^2}$, Using the product rule: $y = \frac{1}{x^2}\ln(5x)$ (1 mark)

$$\Rightarrow \frac{dy}{dx} = -\frac{2}{x^3}\ln(5x) + \frac{1}{x^2} \times \frac{1}{x}$$ (2 marks)

$$= \frac{1 - 2\ln(5x)}{x^3}$$ (1 mark)

$$= \frac{1 - \ln(25x^2)}{x^3}$$ (1 mark)

[5 marks]

c) $f(x, y) = x^2 + y^2 - 2xy$

(i) $\frac{\partial f}{\partial x} = 2x - 2y$ $\frac{\partial f}{\partial y} = 2y - 2x$ [2 marks]

(ii) $x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = x(2x - 2y) + y(2y - 2x)$

$= 2x^2 + 2y^2 - 4xy$ (1 mark)

$= 2(x^2 + y^2 - 2xy)$ (1 mark)

$= 2f(x, y)$ (1 mark)

[3 marks]

Total 25 marks

SECTION B

(MODULE 2)

Question 3

(a) (i) $$\frac{1}{(2r-1)(2r+1)} \equiv \frac{A}{2r-1} + \frac{B}{2r+1}$$

$\Rightarrow$ $1 = A(2r+1) + B(2r-1)$ (1 mark)

$\Rightarrow$ $0 = 2A + 2B$ and $A - B = 1$ (2 marks)

$\Rightarrow$ $A = \frac{1}{2}$ and $B = -\frac{1}{2}$ (2 marks)

[5 marks]

(ii) $$S = \sum_{r=1}^{n} \frac{1}{(2r-1)(2r+1)} = \sum_{r=1}^{n} \frac{1}{2}\left(\frac{1}{2r-1} - \frac{1}{2r+1}\right)$$ (1 mark)

$$= \frac{1}{2}\left(\frac{1}{1} - \frac{1}{3}\right) + \frac{1}{2}\left(\frac{1}{3} - \frac{1}{5}\right) + \frac{1}{2}\left(\frac{1}{5} - \frac{1}{7}\right) + \ldots + \frac{1}{2}\left(\frac{1}{2n-1} - \frac{1}{2n+1}\right)$$

(3 marks)

$$= \frac{1}{2}\left(1 - \frac{1}{2n+1}\right)$$ (1 mark)

[5 marks]

(iii) As $n \to \infty$, $\frac{1}{2n+1} \to 0$ (2 marks)

Hence $S_\infty = \frac{1}{2}$ (1 mark)

[3 marks]

(b) (i) $S = 1(2) + 2(5) + 3(8) + \ldots$

In each term, 1st factor is in the natural sequence and the second factor differs by 3 (1 mark)

$\Rightarrow$ the r^{th} term is $r(3r - 1)$ (1 mark)

[2 marks]

(ii) $$S_n = \sum_{r=1}^{n} r(3r-1)$$

for $n = 1$ $\quad S_1 = \sum_{r=1}^{1} r(3r-1) = 1 \times 2 = 2$

and $1^2(1+1) = 1 \times 2 = 2$ (1 mark)

hence, $S_n = n^2(n+1)$ is true for $n = 1$ (1 mark)

Assume $S_n = n^2(n+1)$ for $n = k \in \mathbb{N}$ (1 mark)

that is, $S_k = k^2(k+1)$ (1 mark)

Then, $S_{k+1} = \sum_{r=1}^{k+1} r(3r-1) = S_k + (k+1)(3k+2)$ (1 mark)

$= k^2(k+1) + (k+1)(3k+2)$ (1 mark)

$= (k+1)\left[k^2 + 3k + 2\right]$ (1 mark)

$\Rightarrow \quad S_{k+1} = (k+1)\left[(k+1)(k+2)\right]$

$= (k+1)^2\left[(k+1)+1\right]$ (1 mark)

$\Rightarrow$ true for $n = k + 1$ whenever it is assumed true for $n = k$, (1 mark)

$\Rightarrow$ true for all $n \in N$

$\Rightarrow \quad S_n = n^2(n+1) \; n \in \mathbb{N}$. (1 mark)

[10 marks]

Total 25 marks

Question 4

(a) (i) Let $S \equiv \frac{1}{2}+\frac{1}{2^4}+\frac{1}{2^7}+\frac{1}{2^{10}}+\ldots$

$$\frac{\frac{1}{2^4}}{\frac{1}{2}} = \frac{\frac{1}{2^7}}{\frac{1}{2^4}}$$ (1 mark)

$$= \frac{1}{2^3}$$ (1 mark)

$\therefore$ S is geometric with common ratio $r = \frac{1}{2^3}$ (1 mark)

[3 marks]

(ii) $$S_n = \frac{\frac{1}{2}\left[1-\left(\frac{1}{2}\right)^{3n}\right]}{1-\left(\frac{1}{2}\right)^3}$$ (1 mark)

$$= \frac{\frac{1}{2}\left[1-\frac{1}{2^{3n}}\right]}{1-\frac{1}{8}}$$ (1 mark)

$$= \frac{1}{2}\times\frac{8}{7}\left[1-\frac{1}{2^{3n}}\right]$$ (1 mark)

$$= \frac{4}{7}\left[1-\frac{1}{2^{3n}}\right]$$ (1 mark)

[4 marks]

(b) (i) $f(x) = \cos 2x \quad \Rightarrow \quad f^{1}(x) = -2\sin 2x$ (1 mark)

$\Rightarrow \quad f^{11}(x) = -4\cos 2x$ (1 mark)

$\Rightarrow \quad f^{111}(x) = 8\sin 2x$ (1 mark)

$\Rightarrow \quad f^{iv}(x) = 16\cos 2x$ (1 mark)

so, $f(0)=1,\ f^{1}(0)=0,\ f^{11}(x)=-4,\ f^{111}(0)=0,\ f^{iv}(0)=16$ (1 mark)

Hence, by Maclaurin's Theorem,

$\cos 2x = 1 - \dfrac{4x^2}{2!} + \dfrac{16x^4}{4!} - \ldots..$ (1 mark)

$= 1 - 2x^2 + \dfrac{2}{3}x^4 \ldots\ldots$ (1 mark)

[7 marks]

(c) (i) $\sqrt{\left(\dfrac{1+x}{1-x}\right)}$

$= (1+x)^{½}\ (1-x)^{-½}$ (1 mark)

$= \left(1+\dfrac{1}{2}x-\dfrac{1}{8}x^2+\dfrac{1}{16}x^3\ldots\right)\left(1+\dfrac{1}{2}x+\dfrac{3}{8}x^2+\dfrac{5}{16}x^3\ldots\right)$ (3 marks)

$= 1 + x + \dfrac{1}{2}x^2 + \dfrac{1}{2}x^3$ (2 marks)

for $-1 < x < 1$ (1 mark)

[7 marks]

(ii) $\sqrt{\dfrac{1.02}{0.98}} = \sqrt{\dfrac{102}{98}} = \dfrac{1}{7}\sqrt{51}$ (1 mark)

$\sqrt{51} = 7\sqrt{\dfrac{1+x}{1-x}}$ where $x = 0.02$ (1 mark)

$\Rightarrow \sqrt{51} = 7\left\{1+0.02+\dfrac{1}{2}(0.02)^2+\dfrac{1}{2}(0.02)^3\right\}$ (1 mark)

$= 7.14141$ (5 d.p.) (1 mark)

[4 marks]

Total 25 marks

SECTION C

(MODULE 3)

Question 5

(a) (i) P (First card drawn has even number) $= \frac{5}{10} = \frac{1}{2}$ (1 mark)

P (Second card drawn has even number) $= \frac{4}{9}$ (2 marks)

$\therefore$ P (Both cards have even numbers) $= \left(\frac{1}{2}\right)\left(\frac{4}{9}\right)$

$= \frac{2}{9}$ (1 mark)

[4 marks]

(ii) P (Both cards have odd numbers) $= \frac{2}{9}$ (1 mark)

P (One card has odd and the other has even i.e. both cards do not have odd or do not have even numbers) $= 1 - 2\left(\frac{2}{9}\right)$ (2 marks)

$= \frac{5}{9}$ (1 mark)

[4 marks]

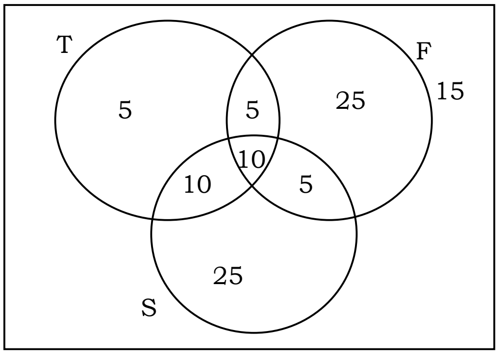
(b) (i) $\frac{82}{150} = 0.547$ [2 marks]

(ii) $\frac{39}{150} + \frac{75}{150} = 0.76$ [4 marks]

(iii) $\frac{82}{150} + \frac{39}{150} - \frac{27}{150} = 0.267$ [3 marks]

(c) Let T, S and F represent respectively the customers purchasing tools, seeds and fertilizer.

(i) One mark for any two correct numbers [4 marks]



Venn diagram

(ii) a) 5 (1 mark)
b) 10 (1 mark)
c) 5 (1 mark)
d) 15 (1 mark)

[4 marks]

Total 25 marks

Question 6

(a) $\begin{vmatrix} 5 & x & 3 \\ x+2 & 2 & 1 \\ -3 & 2 & x \end{vmatrix} = 0$

$5\,(2x-2) - x\,(x^2+2x+3) + 3\,(2x+4+6) = 0$ (3 marks)

$x^3 + 2x^2 - 13x - 20 = 0$ (1 mark)

Subs $x = -4$, $(-4)^3 + 2\,(-4)^2 - 13(-4) - 20 = 0$

$(x+4)\,(x^2 - 2x - 5) = 0$ (2 marks)

$x = -4$

$x = \dfrac{2 \pm \sqrt{24}}{2}$

$x = 1 \pm \sqrt{6}$ (2 marks)

[8 marks]

Alternatively

$\begin{vmatrix} 5 & x & 3 \\ x+2 & 2 & 1 \\ -3 & 2 & x \end{vmatrix} = 0 \Rightarrow \begin{vmatrix} 4+x & x+4 & x+4 \\ x+2 & 2 & 1 \\ -3 & 2 & x \end{vmatrix} = 0$ (Add rows 2 and 3 to row 1)

$\Rightarrow (x+4)\begin{vmatrix} 1 & 1 & 1 \\ x+2 & 2 & 1 \\ -3 & 2 & x \end{vmatrix} = 0 \Rightarrow (x+4)\begin{vmatrix} 0 & 1 & 0 \\ x & 2 & -1 \\ -5 & 2 & x-2 \end{vmatrix} = 0$

(subtract columns 2 from Columns 1 and 3).

$\Rightarrow (x+4)x - (x^2 - 2x - 5) = 0 \Rightarrow x = -4$ or $1 \pm \sqrt{6}$ [8 marks]

(b) $y \tan x \dfrac{dy}{dx} = (4 + y^2)\sec^2 x$ (1 mark)

$\dfrac{y}{4+y^2}\dfrac{dy}{dx} = \dfrac{\sec^2 x}{\tan x}$

$\dfrac{y\,dy}{4+y^2} = \dfrac{\sec^2 x\,dx}{\tan x}$

$\int \dfrac{y\,dy}{4+y^2} = \int \dfrac{\sec^2 x\,dx}{\tan x}$ (1 mark)

$\frac{1}{2}\ln(4 + y^2) = \ln|\tan x| + c$ (2 marks)

[4 marks]

(c) (i) $y = u\cos 3x + v\sin 3x$

$\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = -3u\sin 3x + 3v\cos 3x$ (2 marks)

$\Rightarrow \frac{\mathrm{d}^2y}{\mathrm{d}x^2} = -9u\cos 3x - 9v\sin 3x$ (2 marks)

so, $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 4\frac{\mathrm{d}y}{\mathrm{d}x} + 3y = -30\sin 3x$

$\Rightarrow -(6v - 12u)\sin 3x + (-6u + 12v)\cos 3x = -30\sin 3x$ (1 mark)

$\Rightarrow 2u + v = 5$ and $u = 2v$ (1 mark)

$\Rightarrow u = 2$ and $v = 1$ (2 marks)

[8 marks]

(ii) the auxiliary equation of the different equation is

$k^2 + 4k + 3 = 0$ (1 mark)

$\Rightarrow (k + 3)(k + 1) = 0$

$\Rightarrow k = -3$ or -1 (2 marks)

$\Rightarrow$ the complementary function is

$y = A\mathrm{e}^{-x} + B\mathrm{e}^{-3x};$ where A, B are constants (1 mark)

General solution is $y = A\mathrm{e}^{-x} + B\mathrm{e}^{-3x} + \sin 3x + 2\cos 3x$ (1 mark)

[5 marks]

Total 25 marks

FORM TP 02234032/SPEC

C A R I B B E A N   E X A M I N A T I O N S   C O U N C I L

ADVANCED PROFICIENCY EXAMINATION

PURE MATHEMATICS

UNIT 2 – Paper 032

COMPLEX NUMBERS, ANALYSIS AND MATRICES

SPECIMEN PAPER

1 hour 30 minutes

The examination paper consists of **THREE** sections: Module 1, Module 2 and Module 3.

Each section consists of 1 question.
The maximum mark for each Module is 20.
The maximum mark for this examination is 60.
This examination consists of 4 printed pages.

READ THE FOLLOWING INSTRUCTIONS CAREFULLY.

1. **DO NOT** open this examination paper until instructed to do so.
2. Answer **ALL** questions from the **THREE** sections.
3. Write your solutions, with full working, in the answer booklet provided.
4. Unless otherwise stated in the question, any numerical answer that is not exact **MUST** be written correct to **THREE** significant figures.

Examination Materials Permitted

Graph paper (provided)
Mathematical formulae and tables (provided) – **Revised 2010**
Mathematical instruments
Silent, non-programmable electronic calculator

DO NOT TURN THIS PAGE UNTIL YOU ARE TOLD TO DO SO

SECTION A (MODULE 1)

Answer this question.

1. (a) (i) Given that $z = 8 + (8\sqrt{3})\text{i}$, find the modulus and argument of z. **[3 marks]**

(ii) **Using de Moivre's theorem**, show that z^3 is real, stating the value of z^3. **[2 marks]**

(b) A complex number is represented by the point, P, in the Argand diagram.

(i) Given that $|z - 6| = |z|$ show that the locus of P is $x = 3$. **[2 marks]**

(ii) Determine the TWO complex numbers which satisfy both

$$|z - 6| = |z| \text{ and } |z - 3 - 4\mathbf{i}| = 5.$$ **[5 marks]**

(c) Given $I_n = \int_0^8 x^n (8-x)^{1/x}\, \mathrm{d}x, \quad n \geq 0,$ show that

$$I_n = \frac{24n}{3n+4} I_{n-1}, \; n \geq 1.$$ **[8 marks]**

Total 20 marks

SECTION B (MODULE 2)

Answer this question.

2. (a) (i) Show that $(r+1)^3 - (r-1)^3 = 6r^2 + 2$. **[2 marks]**

(ii) Hence, show that $\sum_{r=1}^{n} r^2 = \frac{n}{6}(n+1)(2n+1)$. **[5 marks]**

(iii) Show that $\sum_{r=n}^{2n} r^2 = \frac{n}{6}(n+1)(an+b)$, where a and b are unknown constants.

[4 marks]

(b) The displacement, x metres, of a particle at time, t, seconds is given by the differential equation

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + x + \cos x = 0.$$

When $t = 0$, $x = 0$ and $\frac{\mathrm{d}x}{\mathrm{d}t} = 0.5$.

Find a Taylor series solution for x in ascending powers of t, up to and including the term in t^3. **[5 marks]**

(c) Given that α is the only real root of the equation

$$x^3 - x^2 - 6 = 0,$$

(i) Show that $2.2 < \alpha < 2.3$. **[2 marks]**

(ii) Use linear interpolation once on the interval [2.2, 2.3] to find another approximation to α, giving your answer to 3 decimal places. **[2 marks]**

Total 20 marks

SECTION C (MODULE 3)

Answer this question.

3. (a) Three identical cans of cola, two identical cans of tea and two identical cans of orange juice are arranged in a row.

Calculate the number of arrangements if the first and last cans in the row are of the same type of drink. **[3 marks]**

(b) Kris takes her dog for a walk every day. The probability that they go to the park on any day is 0.6. If they go to the park, there is a probability of 0.35 that the dog will bark. If they do not go to the park, there is probability of 0.75 that the dog will bark.

Find the probability that the dog barks on any particular day. **[2 marks]**

(c) A committee of six people, which must consist of at least 4 men and at least one woman, is to be chosen from 10 men and 9 women, including Albert and Tracey.

Find the number of possible committees that include either Albert or Tracey but not both. **[3 marks]**

(d) A and B are two matrices such that

$$A = \begin{pmatrix} 1 & -1 & 3 \\ 2 & 1 & 4 \\ 0 & 1 & 1 \end{pmatrix} \text{ and } B = \begin{pmatrix} -3 & 4 & -7 \\ -2 & 1 & 2 \\ 2 & -1 & 3 \end{pmatrix}$$

(i) Find AB. **[2 marks]**

(ii) Deduce A^{-1}. **[2 marks]**

(iii) Given that $B^{-1} = \frac{1}{5}\begin{pmatrix} 1 & -1 & 3 \\ 2 & 1 & 4 \\ 0 & 1 & 1 \end{pmatrix}$, prove that $(AB)^{-1} = B^{-1}A^{-1}$. **[2 marks]**

(e) Find the general solution of the differential equation

$$\frac{dy}{dx} + y\cot x = \sin x.$$ **[6 marks]**

Total 20 marks

END OF TEST

IF YOU FINISH BEFORE TIME IS CALLED, CHECK YOUR WORK ON THIS TEST.

C A R I B B E A N   E X A M I N A T I O N S   C O U N C I L

ADVANCED PROFICIENCY EXAMINATION

PURE MATHEMATICS

UNIT 2

COMPLEX NUMBERS, ANALYSIS AND MATRICES

SPECIMEN PAPER

PAPER 032

SOLUTIONS AND MARK SCHEMES

Question	Details	Marks
1 (a) (i)	$\lvert z\rvert = \sqrt{8^2 + \left(8\sqrt{3}\right)^2} = 16$	1
	arg (z) = $\pi - \tan^{-1}\left(\sqrt{3}\right) = \frac{2\pi}{3}$ (1) (1)	1 1
(ii)	$z^3 = 16^3\,(\cos 2\pi/3 + i \sin 2\pi/3)^3$	1
	$= 16^3\,(\cos 2\pi + i \sin 2\pi) = 4096$	1
(b) (i)	$(x-6)^2 + y^2 = x^2 + y^2$	1
	$x = 3$	1
(ii)	$\lvert z - 3 - 4i\rvert = 5$ is a circle with centre (3, 4) and radius 5	1
	Subs $x = 3$ gives $(3-3)^2 + (y-4)^2 = 5^2$	1
	$y^2 - 8y - 9 = 0 \quad (y-9)(y+1) = 0 \quad y = 9, -1$	1
	z = 3 + 9i, 3 – i (1) (1)	1 1
(c)	$I_n = \frac{3}{4}\left[x^n\,(8-x)^{4/3}\right] + \frac{3}{4}\int nx^{n-1}\,(8-x)^{4/3}\,dx$ (1) (1)	1 1
	$= 0 + \frac{3}{4}\int nx^{n-1}\,(8-x)(8-x)^{1/3}\,dx$ (1) (1)	1 1
	$= \frac{3}{4}\int nx^{n-1}\,8\,(8-x)^{1/3}\,dx - \frac{3}{4}\int nx^{n-1}\,x(8-x)^{1/3}\,dx$ (1) (1)	1 1
	$I_n = 6n\,I_{n-1} - \frac{3n}{4}\,I_n$	1
	$I_n = \frac{24n}{3n+4}\,I_{n-1}$	1
		[20]
	S. O. (A) 7, 8, 11, 12, (C) 10	

Question	Details	Marks
2 (a)(i)	$r^3 + 3r^2 + 3r + 1 - (r^3 - 3r^2 + 3r - 1)$	1
	$= 6r^2 + 2$	1
(ii)	$r = 1$: $2^3 - 0^3 = 6(1)^2 + 2$ $r = 2$: $3^3 - 1^3 = 6(2)^2 + 2$ …………………………….. ……………………………..	1
	$r = n$: $(n + 1)^3 - (n - 1)^3 = 6n^2 + 2$	1
	summing gives $(n + 1)^3 + n^3 - 1 = 6\sum_1^n r^2 + 2n$ (1) (1)	1 1
	$\sum_{r=11}^{n} r^2 = \frac{n}{6}(n+1)(2n+1)$	1
(iii)	$\sum_{r=n}^{2n} r^2 = \sum_{r=1}^{2n} r^2 - \sum_{r=1}^{n-1} r^2$	1
	$= \frac{2n}{6}(2n+1)(4n+1) - \frac{n-1}{6}(n)(2n-1)$ (1) (1)	1 1
	$= \frac{n}{6}(n+1)(14n+1)$	1
(b)	$f^{11}(t) = -x - \cos x$ $f^{11}(0) = -1$ (1) (1)	1 1
	$f^{111}(t) = (-1 + \sin x)\frac{dx}{dt}$ $f^{111}(0) = -0.5$ (1) (1)	1 1
	$f(t) = \frac{1}{2}t - \frac{1}{2}t^2 - \frac{1}{12}t^3 + \ldots$	1
(c) (i)	$f(2.2) = -0.192$ $f(2.3) = 0.877$ By IVT and continuity root exists (1) (1)	1 1
(ii)	$\frac{\alpha - 2.2}{0.192} = \frac{2.3 - \alpha}{0.877}$	1
	$\alpha = 2.218$	1
		[20]

S. O. (B) 1, 2, 4, 9, (D) 1, 3		
Question	**Details**	**Marks**
3 (a)	Ends cola: $\dfrac{5!}{2!2!} = 30$ ways	1
	Ends green tea: $\dfrac{5!}{3!2!} = 10$ ways	1
	Ends orange juice: $\dfrac{5!}{3!2!} = 10$ ways Total = 30 + 10 + 10 = 50 ways	1
(b)	P(bark) = P(park & bark) + P(no park & bark)	1
	= (0.6)(0.35) + (0.4)(0.75) = 0.51	1
(c)	Albert not Tracey = (9 C 3) × (8 C 2) + (9 C 4) × (8 C 1) = 3360	1
	Tracey not Albert = (9 C 4) × (8 C 1) + (9 C 5) = 1134	1
	# of selections = 3360 + 1134 = 4494	1
(d) (i)	$AB = \begin{pmatrix}1 & -1 & 3\\ 2 & 1 & 4\\ 0 & 1 & 1\end{pmatrix} \times \begin{pmatrix}-3 & 4 & -7\\ -2 & 1 & 2\\ 2 & -1 & 3\end{pmatrix} = \underset{(1)}{\begin{pmatrix}5 & 0 & 0\\ 0 & 5 & 0\\ 0 & 0 & 5\end{pmatrix}} = \underset{(1)}{5I}$	1 1
(ii)	$AA^{-1}B = 5A^{-1}$	1
	$A^{-1} = \dfrac{1}{5}\begin{pmatrix}-3 & 4 & -7\\ -2 & 1 & 2\\ 2 & -1 & 3\end{pmatrix}$	1
(iii)	$(AB) = 5I$ $(AB)(AB)^{-1} = 5(AB)^{-1}$	
	$(AB)^{-1} = \dfrac{1}{5}\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}$	1
	$B^{-1}A^{-1} = \dfrac{1}{5}\begin{pmatrix}1 & -1 & 3\\ 2 & 1 & 4\\ 0 & 1 & 1\end{pmatrix} \times \dfrac{1}{5}\begin{pmatrix}-3 & 4 & -7\\ -2 & 1 & 2\\ 2 & -1 & 3\end{pmatrix}$	
	$= \dfrac{1}{5}\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}$	1

Question	**Details**	**Marks**
3 (e)	$I = e^{\int \cot x \, dx}$	1
	$= \sin x$	1
	$\sin x \dfrac{dy}{dx} + y \cos x = \sin^2 x$	1
	$\int \dfrac{d}{dx}(y \sin x)\, dx = \dfrac{1}{2} \int (1 - \cos 2x)\, dx$ (1) (1)	1 1
	$y \sin x = \frac{1}{2}\left(x - \frac{1}{2} \sin 2x\right) + C$	1
		[20]
S. O. (A) 3, 4, 16, (B) 1, 2, 7, (C) 1		